GRADUATED EXERCISES

IN

PLANE TRIGONOMETRY,

COMPILED AND ARRANGED BY

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PREFACE.

THE following collection of Examples is intended for the use of schools and junior students generally. The exercises are carefully graduated in difficulty, and by the constant repetition of examples on the earlier parts of Trigonometry the student will be prevented from forgetting them whilst occupied with more advanced work. Those Examples which are not original have been selected from various College and University Examination Papers, and will, we believe, be very useful in accustoming the student to the kind of work he will especially need. The knowledge of the determinant notation is now so common, that we feel we need not apologize for introducing some examples expressed in this form. A few notes are scattered through the book, calling attention expressly to theorems or modes of solution of problems which are likely to be useful. Every care has been taken to verify the answers given. A table of abbreviations is prefixed to the book to avoid repeated definitions of the same symbols.

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CONTENTS.

LIST OF ABBREVIATIONS.

The following abbreviations are used in this book in problems relating to triangles:

a, b, c denote the lengths of the sides BC, CA, AB respectively.

s denotes the semi-sum of the sides.

Δ ,, the area.

R ,, the radius of the circumscribing circle.

r ,, ,, inscribed circle.

r_a, r_b, r_c denote the radii of the escribed circles touching BC, CA, AB respectively.

The straight lines drawn from the angles perpendicular to the opposite sides are spoken of as the *perpendiculars*. Their lengths are denoted by p_a, p_b, p_c respectively, and their point of intersection is called the *orthocentre*.

The straight lines drawn from the angles to bisect the opposite sides are called the *bisectors* or *medians*. These must not be confounded with the bisectors of the angles of the triangle.

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IN

PLANE TRIGONOMETRY.

EXERCISE I.

1. FIND the arc subtending an angle of one degree at the centre of a circle whose radius is 4000 miles.

2. What is the circular measure of 18°, $\frac{50}{\pi}$ degrees, 10°, and of $\frac{120}{\pi}$ degrees?

3. Find the measure in degrees, minutes, and seconds, of the angle subtended by an arc equal to the radius; assuming that $\pi = 3{\cdot}14159$.

4. How many grades, minutes and seconds are there in $54^\circ\ 13'\ 21''$?

5. An angle is such that the difference of the reciprocals of the numbers of degrees and grades in it is equal to its circular measure divided by 2π; find the angle.

6. If the three numbers which express A, B, and C, the angles of a triangle, are all equal, the unit of measurement of A being a degree, of B a grade, and of C an angle equal to the sum of a degree and a grade; express each of the angles in circular measure.

EXERCISE II.

1. Find the magnitude of an angle of a regular polygon of 48 sides, in sexagesimal measure.

2. Express $\frac{\pi}{25}$ and $\frac{\pi}{60}$ in degrees; and find the number of grades in $49^\circ\ 43'\ 30''$.

3. What are the values of $\sin\theta - \cos\theta$ when θ has the values $0, \frac{\pi}{4}, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$?

4. Prove that

(1) $$\sin\theta\cos\theta = \frac{\tan\theta}{1+\tan^2\theta} = \frac{\cot\theta}{1+\cot^2\theta}.$$

(2) $$(1-\operatorname{vers}\theta)^2 = \frac{\cot^2\theta}{\operatorname{cosec}^2\theta}.$$

(3) $$\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta\operatorname{cosec}^2\theta.$$

5. If $\tan A = \frac{1}{2}$ find the values of the following expressions:

(1) $\cos A - \sin A$,
(2) $\cos^2 A - \sin^2 A$,
(3) $\operatorname{cosec}^2 A - \sec^2 A$,
(4) $\cot^2 A + \sin^2 A$.

EXERCISE III.

1. Find the values of the following expressions:

(1) $\tan 690^\circ - \sin 570^\circ$,

(2) $\operatorname{vers}\frac{\pi}{4} + \operatorname{vers}\frac{\pi}{2} + \operatorname{vers}\frac{3\pi}{4} + \operatorname{vers}\pi$,

(3) $\sin 150^g - \cos 50^g$.

2. Shew that the angle subtended at the centre of a circle of radius r by an arc a is $\frac{ma}{r}$ where m depends solely on the unit of angular measurement employed. Determine m when the unit is an angle of 18°.

3. Find the length of the circumference of a circle of 6 inches radius. Find the angle which an arc of the same length would subtend at the centre of a circle of 15 inches radius.

4. Express the following in terms of ratios of angles less than a right angle:

(1) $2\sin 510^\circ + \tan 703^\circ \tan 253^\circ - \operatorname{cosec} 236^\circ$,

(2) $2\cos^2 135^\circ + \sec 275^\circ - \dfrac{\tan 225^\circ \cot 210^\circ}{\sqrt{3}}$,

(3) $\cos\dfrac{10\pi}{9} - \sin\dfrac{23\pi}{6} + \dfrac{3}{2}\cot^2\dfrac{5\pi}{3}$.

5. Express the angle of a regular polygon of 32 sides in degrees, grades, and circular measure.

EXERCISE IV.

1. The apparent angular diameter of the sun is half a degree. A planet is seen to cross its disc in a straight line at a distance from the centre equal to three-fifths of the radius. Prove that the angle subtended at the earth by the part of the planet's path projected on the sun is very nearly $\frac{\pi}{450}$.

2. If $\sin A = \frac{4}{5}$ and $\sin B = \frac{5}{13}$ find the values of

(1) $\tan(A+B)$,

(2) $\cot(A-B)$,

(3) $\sin 2A$,

(4) $\sin(A+B)$,

(5) $\cos(A-B)$.

3. Find the values of the expressions in the last example, when $\tan A = \frac{4}{3}$, $\tan B = \frac{5}{12}$, and explain why the results are not the same as those obtained before.

4. Prove that

(1) $\cos A = \cos^4 \dfrac{A}{2} - \sin^4 \dfrac{A}{2}$,

(2) $\sin A = 4 \sin \dfrac{A}{4} \cos \dfrac{A}{4} \cos \dfrac{A}{2}$.

5. Find approximately the number of seconds which an arc of 1 metre subtends at the centre of a circle whose radius is 4000 metres.

EXERCISE V.

1. Prove that

(1) $\tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right) + \cot\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right) = 2 \sec\theta$,

(2) $\sec\theta = 1 + \tan\theta \tan\dfrac{\theta}{2}$.

2. How many grades are there in the angle which is the arithmetical mean between the angle of a regular pentagon and that of a regular quindecagon? Express the harmonic mean of those angles in circular measure.

3. Find the value of the products:

(1) $16 \sin 3^\circ 45' \cos 3^\circ 45' \cos 7^\circ 30' \cos 15^\circ \cos 30^\circ$.

(2) $(\sin 15^\circ + \sin 75^\circ)\ (\cos 105^\circ + \cos 15^\circ)$.

4. Shew that

$$\sin^2 (45^\circ - A),\ \sin^2 45^\circ, \text{ and } \sin^2 (45^\circ + A)$$

are in Arithmetical Progression; and that

$$\sec^2 (45^\circ - A),\ \sec^2 45^\circ, \text{ and } \sec^2 (45^\circ + A)$$

are in Harmonical Progression.

5. Prove that

(1) $$\sin 2\theta = \frac{1 - \cot^2\left(\frac{\pi}{4} + \theta\right)}{1 + \cot^2\left(\frac{\pi}{4} + \theta\right)},$$

(2) $$\sec 2\phi\,(1 + \sec\phi \cos 3\phi) = 2.$$

Transformation of Formulæ.

A Beginner is frequently perplexed how to set about examples of this nature. No general rules can be given, but it is advisable that he should commence with the more complicated portion of the identity, as it is usually easier to arrive at a given result by simplifying the formula than by the reverse process. We especially caution him not to work both sides of the identity at once, so as to arrive at the result $1 = 1$ or something equally self-evident: one portion of the identity must be transformed into the other. The beginner must of course remember the fundamental formulæ for the expansion of $\sin (A + B)$, &c., but no dexterity in transformations can be acquired without

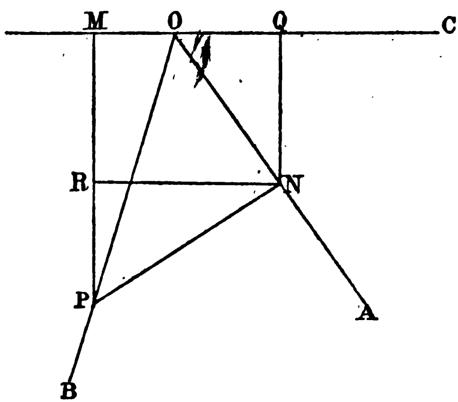
the constant use of the formulæ for the addition and subtraction of sines and cosines—as, for example,

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2},$$

and of those for the reverse process of breaking up products of sines and cosines into sums and differences—as, for example,

$$\sin A \cos B = \tfrac{1}{2} \{\sin (A+B) + \sin (A-B)\}.$$

The student should be able to prove the fundamental formulæ when the angles have any specified values, however large. We give proofs of the expansions of $\sin (A-B)$ and $\tan (A-B)$ when A lies in the fourth quadrant, and $A-B$ in the third; the other formulæ may be proved in like manner as an exercise.



Let the angle COA, measured in the positive direction, equal A and $AOB = B$; then $COB = A - B$.

In OB take any point P, and draw PN perpendicular to OA, PM perpendicular to CO *produced*, NQ and NR perpendicular to OC and PM.

Then the angle

$$NPR = 90^\circ - PMR = OMR = COA = 360^\circ - A,$$

$$\begin{aligned}
\therefore \sin(A-B) &= \sin(180^\circ + POM) \\
&= -\sin POM \\
&= -\frac{PM}{OP} \\
&= -\frac{NQ}{OP} - \frac{PR}{OP} \\
&= -\frac{NQ}{ON}\cdot\frac{ON}{OP} - \frac{PR}{PN}\cdot\frac{PN}{OP} \\
&= -\sin(360^\circ - A)\cos B - \cos(360^\circ - A)\sin B \\
&= \sin A\cos B - \cos A\sin B.
\end{aligned}$$

Also
$$\begin{aligned}
\tan(A-B) &= \tan(180^\circ + POM) \\
&= \tan POM \\
&= \frac{PM}{MO} \\
&= \frac{PR+NQ}{RN-OQ} \\
&= \frac{\dfrac{PR}{OQ} + \dfrac{NQ}{OQ}}{\dfrac{RN}{PR}\cdot\dfrac{PR}{OQ} - 1}
\end{aligned}$$

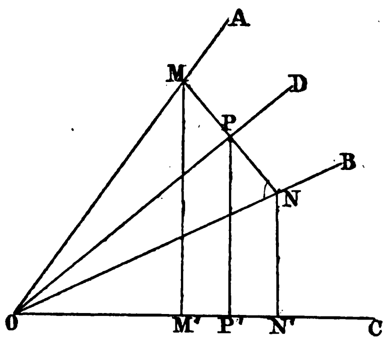
Since the angles NPR and QON are equal, the right-angled triangles NPR and QON are similar; thus we see that $\dfrac{PR}{PN} = \dfrac{OQ}{ON}$;

$$\therefore \frac{PR}{OQ} = \frac{PN}{ON} = \tan B.$$

Thus $$\tan(A - B) = \frac{\tan B + \tan(360° - A)}{\tan(360° - A)\tan B - 1}$$
$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

The formulæ for the addition and subtraction of sines and cosines may be easily deduced from the four fundamental formulæ; but they may also be proved independently. As a specimen of the method, we will shew that

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$



Let the angle COA be A, and the angle COB be B. Bisect the angle AOB by the straight line OD; then

$$COD = \frac{A+B}{2} \text{ and } BOD = \frac{A-B}{2}.$$

In OD take any point P, and draw MPN perpendicular to OD. Also draw MM', PP', and NN' perpendicular to OC.

Then we can easily shew that $OM = ON$, $MP = PN$, and therefore that $OM' + ON' = 2\,OP'$;

$$\therefore \cos A + \cos B = \frac{OM'}{OM} + \frac{ON'}{ON}$$

$$= \frac{OM' + ON'}{ON}$$

$$= 2\,\frac{OP'}{ON}$$

$$= 2\,\frac{OP'}{OP} \cdot \frac{OP}{ON}$$

$$= 2 \cos COD \cos BOD$$

$$= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}.$$

EXERCISE VI.

1. Find the number of degrees in the angle whose circular measure is approximately ·0314159.

2. Prove that

(1) $\cot^2 A \cos^2 A = \cot^2 A - \cos^2 A$,

(2) $\dfrac{\cot A}{\cot 2A} = 1 + \sec 2A$,

(3) $\tan^2 A + \cot^2 A = 2 + 4 \cot^2 2A$.

3. Simplify

(1) $\dfrac{\cos \alpha - \cos 5\alpha}{\sin \alpha + \sin 5\alpha}$,

(2) $\dfrac{\sin 5\theta - \sin 3\theta}{\cos 5\theta + \cos 3\theta}$.

4. Express the sine, cosine, and cotangent, of each of the angles 1962°, 2376°, 2844°, in terms of trigonometrical ratios of angles between 0 and 45°.

5. A railway train is travelling on a circle of three-quarters of a mile radius at the rate of thirty miles an hour; through what angle has it turned in a quarter of a minute?

EXERCISE VII.

1. Prove that

(1) $$\frac{\text{cosec } 2A}{1+\text{cosec } 2A} = \frac{1+\tan^2 A}{(1+\tan A)^2},$$

(2) $$\frac{\cos 3A + \cos 5A}{\cos 5A + \cos 7A} = \frac{\cos 4A}{\cos 6A},$$

(3) $$\sqrt{\text{cosec}^2 x - 1}\,\sqrt{2 \text{ vers } x - \text{vers}^2 x} = \cos x.$$

2. If $\tan A = \frac{5}{12}$, $\tan B = \frac{20}{21}$, and $\cos C = \frac{21}{29}$, find the various values that $\sin (A+B+C)$ assumes by taking account of the double signs which arise.

3. Prove that

(1) $\tan 50° + \cot 50° = 2 \sec 10°$;

(2) $\tan 50° - \cot 50° = 2 \tan 10°$.

4. Express in each system of measurement the angle between the long and short hands of a watch at a quarter past twelve o'clock.

5. Prove that

(1) $\sin 3A = 4 \sin A \sin (60° + A) \sin (60° - A)$,

(2) $\cos 3A = 4 \cos A \cos (60° + A) \cos (60° - A)$.

EXERCISE VIII.

1. Simplify the expressions:

(1) $\dfrac{\sin 4\theta - \sin 2\theta}{\cos 4\theta + \cos 2\theta} + \dfrac{\sin 4\theta + \sin 2\theta}{\cos 4\theta - \cos 2\theta}$,

(2) $\dfrac{\sin \alpha + \sin 7\alpha}{\cos \alpha - \cos 7\alpha} (1 - \cos 6\alpha)$,

(3) $\dfrac{(\sin 7\phi - \sin 5\phi)(\cos 4\phi - \cos 6\phi)}{(\cos 7\phi + \cos 5\phi)(\sin 4\phi + \sin 6\phi)}$.

2. Shew that

$$2 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta}{2} \cos \frac{\alpha}{2} + \cos^2 \frac{\beta}{2} + \sin^2 \frac{\alpha}{2} + \cos^2 \frac{\alpha + \beta}{2} = 2.$$

3. If $\tan\left(45^\circ - \dfrac{A}{2}\right) = \tan^2 B$,

then

$$\sqrt{\sec A + \tan A} + \sqrt{\sec A - \tan A} = 2 \operatorname{cosec} 2B.$$

4. Prove that

(1) $8 \sin^4 \theta = 3 - 4 \cos 2\theta + \cos 4\theta$,

(2) $\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$.

5. Find the length of an arc of 80° in a circle of four miles radius.

6. Prove that

(1) $\tan (45^\circ + A) - \tan (45^\circ - A) = 2 \tan 2A$,

(2) $\dfrac{\sec A + \tan A}{\sec A - \tan A} = \dfrac{\tan\left(45^\circ + \dfrac{A}{2}\right)}{\tan\left(45^\circ - \dfrac{A}{2}\right)}$.

EXERCISE IX.

1. If
$$\cot A = \frac{2}{\sqrt{5}},$$
find $\sin A$, $\cos A$, $\sin 2A$, and $\tan 2A$; and find $\cos \frac{B}{2}$ and $\tan \frac{B}{2}$, where $\operatorname{cosec} B = \frac{5}{4}$.

2. Prove that

(1) $$\frac{\tan\theta + \cot\theta + 2}{\tan\theta + \cot\theta - 2} = \frac{\sin^2\left(\frac{\pi}{4} + \theta\right)}{\sin^2\left(\frac{\pi}{4} - \theta\right)} = \frac{1 + \sin 2\theta}{1 - \sin 2\theta},$$

(2) $$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \tan 2\theta + \sec 2\theta.$$

3. Shew that

(1) $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$,

(2) $\cos 4A = 8\cos^4 A - 8\cos^2 A + 1$.

4. Given that the number of grades, degrees, and units of circular measure respectively in the three angles of a triangle are as $200 : 360 : 3\pi$; find the angles.

5. Prove that

(1) $\tan 70^\circ + \tan 20^\circ = 2 \operatorname{cosec} 40^\circ$,

(2) $4\sin 195^\circ \cos 165^\circ = 1$.

EXERCISE X.

1. Prove that

(1) $\tan A - \tan\frac{A}{2} = \tan\frac{A}{2}\sec A$,

(2) $\left(\cot\frac{A}{2} - \tan\frac{A}{2}\right)^2 (1 - 2\tan A\cot 2A) = 4$,

(3) $\tan^2\frac{A}{2} + \cot^2\frac{A}{2} - 4\cot^2 A = 2$.

2. If the sexagesimal measure α of one angle be the circular measure of another, and if the tangents of the two angles be also equal, find a general expression for α.

3. Prove that

(1) $\dfrac{\sin 5\theta + \sin 7\theta}{\sin 2\theta(1+2\cos 2\theta)} = 2\cos 3\theta$,

(2) $\cos 6A = 16(\cos^6 A - \sin^6 A) - 15\cos 2A$.

4. Express $\tan 3\theta$ in terms of $\tan\theta$, and from the result deduce the expression for $\sin 3\theta$ in terms of $\sin\theta$.

5. Prove that
$\sin^2(A+B) = \sin 2A\sin 2B + \sin^2(A-B)$.

EXERCISE XI.

1. Solve the equation

$$\cos\theta - \sin\theta = \sqrt{2}\sin\theta.$$

2. Prove the formulæ

(1) $$\sin 3A = (2\cos 2A + 1)\sin A = 3\sin A - 4\sin^3 A,$$

(2) $$\cos 3A = (2\cos 2A - 1)\cos A = 4\cos^3 A - 3\cos A.$$

3. Calculate the distance from the North Pole of the point farthest North (Lat. 82° 20′) reached by the Arctic Expedition of 1875, assuming that the radius of the earth is 3900 miles, and that $\pi = \frac{355}{113}$.

4. Prove that

$$\frac{1 + \sin\theta + \cos\theta}{1 + \sin\theta - \cos\theta} = \cot\frac{\theta}{2}.$$

5. Prove that

(1) $$\cos 4\phi + 2\cos 2\phi + 1 = 4\cos 2\phi\cos^2\phi,$$

(2) $$\cos 2\alpha = 2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta).$$

6. Establish, by means of a figure, the truth of the formula

$$\cos(A - B) = \cos A\cos B + \sin A\sin B,$$

when $A = 240°$ and $B = 30°$.

EXERCISE XII.

1. Solve the equations:

(1) $\sin 2x = \sin x$,

(2) $\sin x + \sin 2x = \cos x + \cos 2x$.

2. Prove that

$$\frac{\sin\theta + \sin\phi + \sin(\theta+\phi)}{\sin\theta + \sin\phi - \sin(\theta+\phi)} = \cot\frac{\theta}{2}\cot\frac{\phi}{2}.$$

3. Shew that

(1) $$\{\cos\beta - \cos\gamma + \sin(\beta-\gamma)\}^2 = 2\{1-\cos(\beta-\gamma)\}\{1-\sin\beta\}\{1-\sin\gamma\},$$

(2) $$(1-\sin\theta)(1-\sin\phi) = \left(\sin\frac{\theta+\phi}{2} - \cos\frac{\theta-\phi}{2}\right)^2.$$

4. Prove that
$$\sin\alpha(\tan^2\alpha+1) = \frac{1+\tan\alpha\tan\frac{\alpha}{2}}{\tan\alpha + 2\cot 2\alpha}.$$

5. Express the Arithmetic and Harmonic means of each pair of the three units of angular measurement (a degree, a grade, and the unit of circular measure) in terms of the third unit.

EXERCISE XIII.

1. Solve the equation
$$\operatorname{cosec} 2x = \cot x.$$

2. Prove that

(1) $\cot A + \tan 2A = \frac{1}{2}\tan 2A \operatorname{cosec}^2 A,$

(2) $4\cos^6\theta + 4\sin^6\theta - 1 = 3\cos^2 2\theta.$

3. Find the value of $\sin 5\theta$ in terms of $\sin\theta$ and from the result deduce an expression for $\cos 5\theta$ in terms of $\cos\theta$.

4. Shew that, in any triangle
$$\cot A(\sin^2 B - \sin^2 C) + \cot B(\sin^2 C - \sin^2 A) + \cot C(\sin^2 A - \sin^2 B) = 0.$$

5. Prove that $\frac{1}{3}(\cos^6\theta+\sin^6\theta)-\frac{1}{4}(\cos^4\theta-\sin^4\theta)^2$ is independent of θ, and find its value.

6. Shew that

(1) $$\operatorname{cosec} 2A+\operatorname{cosec} 2B-\cot 2A-\cot 2B = \sec A \sec B \sin(A+B),$$

(2) $$\cos^2 A\,(1+\operatorname{cosec} 2A-\cot 2A)^2=1+\sin 2A.$$

EXERCISE XIV.

1. Shew that, if $\alpha+\beta+\gamma=\pi$,

$$\cos\alpha \operatorname{cosec}\beta \operatorname{cosec}\gamma+\cos\beta \operatorname{cosec}\gamma \operatorname{cosec}\alpha + \cos\gamma \operatorname{cosec}\alpha \operatorname{cosec}\beta=2.$$

2. Solve the equations

(1) $\sin 3\theta=\cos 7\theta$,

(2) $\sqrt{3}\sin 3\phi=\sin 6\phi$.

3. Given $\tan A=\dfrac{p}{q}$ and $\tan B=\dfrac{m}{n}$,

find (1) $\sin(A-B)$,

(2) $\cos(2A-B)$,

(3) $\tan(A+B)$.

4. Shew that

$$1+\tan A\tan 2A=\tan 2A\cot A-1=\sec 2A.$$

5. Solve the equation

$$\sec A\operatorname{cosec} A=\tan A+\sqrt{3}.$$

6. Shew that

(1) $$\frac{\sin 3\alpha}{\sin \alpha} - \frac{\cos 3\alpha}{\cos \alpha} = 2,$$

(2) $$\frac{\sin 3\alpha}{\sin \alpha} + \frac{\cos 3\alpha}{\cos \alpha} = 4 \cos 2\alpha.$$

EXERCISE XV.

1. Solve the equations

(1) $\sqrt{2} \cos A - \sin A = \cos A.$

(2) $\sec A \operatorname{cosec} A = 1 + \cot A.$

2. If $$\tan \theta = \frac{y \sin \phi}{x + y \cos \phi},$$

prove that $$\tan (\theta - \phi) = \frac{-x \sin \phi}{y + x \cos \phi}.$$

3. Prove that in any triangle

(1) $\sin^2 C = \cos^2 B + \cos^2 A + 2 \cos A \cos B \cos C.$

(2) $$\frac{\sin C + \cos A - \cos B}{\sin C - \sin A - \sin B}$$
$$= \frac{\sin\left(\frac{A}{2} - 45^\circ\right) \cos\left(\frac{B}{2} - 45^\circ\right)}{\sin \frac{A}{2} \sin \frac{B}{2}}$$
$$= \tfrac{1}{2}\left(1 - \cot\frac{A}{2}\right)\left(1 + \cot\frac{B}{2}\right).$$

4. Give a Geometrical proof that
$$\cos 36^\circ = \frac{\sqrt{5} + 1}{4}.$$

5. If $\cos \phi = \frac{5}{13}$ find $\sin\frac{\phi}{2}$, $\sin \phi$, and $\sin 2\phi$.

6. Shew that the value of

$$\frac{\sin\alpha + \cos\beta\sin 3\alpha + \sin 5\alpha}{\cos\alpha + \cos\beta\cos 3\alpha + \cos 5\alpha}$$

is independent of the value of β.

EXERCISE XVI.

1. Find the value of

$$\frac{\sin 5A}{\sin A}\cos 2A,$$

when $$\sin A = \tfrac{4}{5}.$$

2. Solve the equations

(1) $\sin 2\theta + \cos 2\theta = \sin\theta + \cos\theta.$

(2) $\cos 12\theta + 2 = 3\cos 6\theta.$

3. If $\alpha + \beta + \gamma = \dfrac{\pi}{2}$, prove that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta\sin\gamma = 1.$$

4. Find the simplest values of A and B which satisfy

$$\sin(3A - 2B) = 0, \qquad \cos(3B + 2A) = \tfrac{1}{2}.$$

5. Shew that

(1) $$\frac{\cos\alpha}{\cos(\alpha+\beta)\cos(\alpha+\gamma)} - \frac{\sin\beta}{\sin(\beta-\gamma)\cos(\beta+\alpha)} - \frac{\sin\gamma}{\cos(\gamma+\alpha)\sin(\gamma-\beta)} = 0.$$

(2) $$\frac{\cos\alpha}{\sin(\alpha-\beta)\sin(\alpha-\gamma)} + \frac{\cos\beta}{\sin(\beta-\gamma)\sin(\beta-\alpha)} + \frac{\cos\gamma}{\sin(\gamma-\alpha)\sin(\gamma-\beta)} = 0.$$

6. The sides of an isosceles triangle are each 2 inches, and the base angle is 30°. Find the length of the third side, and that of the perpendicular on it.

EXERCISE XVII.

1. Given $\tan A = \frac{21}{20}$ find $\sin\frac{A}{2}$ and $\tan\frac{A}{2}$.

2. In any triangle
$$\cos^2 A = \sin^2 C + \cos^2 B - 2 \sin A \cos B \sin C.$$

3. Solve the equations

(1) $\sin 4\theta - \cos 3\theta + \sin 2\theta - \cos\theta = 0.$

(2) $\sin 3\theta = 2 \sin\theta.$

4. Prove that

(1) $$\frac{\sin\frac{5\pi}{12}}{\sin\frac{\pi}{12}} - \frac{\cos\frac{5\pi}{12}}{\cos\frac{\pi}{12}} = 2\sqrt{3}.$$

(2) $$\frac{\sin\frac{5\pi}{24}}{\cos\frac{\pi}{24}} + \frac{\cos\frac{5\pi}{24}}{\sin\frac{\pi}{24}} = \frac{\sqrt{2}}{\sin\frac{\pi}{12}}.$$

5. Prove that

(1) $$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}.$$

(2) $$(\cos\alpha + \cos\beta)^2 - (\sin\alpha + \sin\beta)^2$$
$$= 4\cos(\alpha+\beta)\cos^2\frac{\alpha-\beta}{2}.$$

6. Shew that

$$\cos\theta+\cos 2\theta+\cos 3\theta+\cos 4\theta=4\cos\frac{\theta}{2}\cos\theta\cos\frac{5\theta}{2}.$$

EXERCISE XVIII.

1. Shew that if $\cos\theta-\sin\theta=\sqrt{2}\sin\theta$,

then $$\cos\theta+\sin\theta=\sqrt{2}\cos\theta.$$

2. If $$\cot\theta\,(1+\sin\theta)=4m$$

and $$\cot\theta\,(1-\sin\theta)=4n,$$

then $$(m^2-n^2)^2=mn.$$

3. Solve the equation

$$\frac{\cos 5\theta+\cos 3\theta}{\sqrt{3}}=\cos\theta.$$

4. If both $\sin A$ and $\cos A$ are given, shew that in general n values and no more may be assigned to $\sin\dfrac{A}{n}$.

5. Prove that

$$\cos(\alpha+\beta)+\sin(\beta-\gamma)-\cos(\gamma+\alpha)$$
$$=2\left(\cos\frac{\alpha+\beta}{2}-\sin\frac{\alpha+\beta}{2}\right)\left(\cos\frac{\gamma+\alpha}{2}-\sin\frac{\gamma+\alpha}{2}\right)\sin\frac{\beta-\gamma}{2}.$$

6. Give a Geometrical proof that

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}.$$

EXERCISE XIX.

1. Solve the equation

$$(\sqrt{5}-1)\cos 3A=2\sin^2 A+\cos 2A.$$

2. Shew that

(1) $\cos 2\alpha + \cos 2\beta - \cos 2\gamma$

$$= 4 \cos (\alpha + \beta) \sin (\beta + \gamma) \sin (\gamma + \alpha) + \cos 2 (\alpha + \beta + \gamma).$$

(2) $\sin (\beta + \gamma - \alpha) + \sin (\gamma + \alpha - \beta) + \sin (\alpha + \beta - \gamma)$

$$= \sin (\alpha + \beta + \gamma) + 4 \sin \alpha \sin \beta \sin \gamma.$$

3. If $4 \sin A \tan (A - B) + 3 \sec A = 4 \tan A \sin A$,

then $$\tan A \tan B = 3.$$

4. Prove from the definition of a logarithm that

$$\log_b a \log_c b \log_a c = 1,$$

and find the characteristics of $\log_7 529$ and $\log_3 \sqrt{\cdot 0214}$.

5. Shew that in any triangle

$\sin^2 A + \sin^2 B + \sin^2 C$

$$= \tfrac{1}{2}(\cot A + \cot B + \cot C)(\sin 2A + \sin 2B + \sin 2C).$$

EXERCISE XX.

1. Eliminate θ between

$$x = a \cos \theta + b \cos 2\theta.$$
$$y = a \sin \theta + b \sin 2\theta.$$

2. Solve the equation

$$\tan^3 \theta - \sec^2 \theta = 4 \tan^2 \theta - 5 \tan \theta.$$

3. If $$\sin \alpha = \cot \alpha \frac{\cos x - \cos \alpha}{\sqrt{2} \cos x - 1},$$

shew that $$\tan \frac{x}{2} \pm \tan \frac{\alpha}{2} \tan \frac{\pi}{8} = 0.$$

4. Shew from a figure that if $\cot \frac{A}{2}$ is expressed in terms of $\cot A$, it will have two values, so related that their product is -1.

5. Prove that, if $\alpha + \beta + \gamma = \frac{\pi}{4}$,

$$(\cos\alpha - \sin\alpha)(\cos\beta + \sin\beta)(\cos\gamma + \sin\gamma) + \sqrt{2}\sin 2\alpha$$
$$= (\cos\beta - \sin\beta)(\cos\gamma + \sin\gamma)(\cos\alpha + \sin\alpha) + \sqrt{2}\sin 2\beta$$
$$= (\cos\gamma - \sin\gamma)(\cos\alpha + \sin\alpha)(\cos\beta + \sin\beta) + \sqrt{2}\sin 2\gamma.$$

6. Prove geometrically that

(1) $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$.

(2) $\cos B - \cos A = 2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$.

EXERCISE XXI.

1. Prove that

(1) $\cos^6\theta + \sin^6\theta = \dfrac{1 + 3\cos^2 2\theta}{4}$.

(2) $\cos^6\theta - \sin^6\theta = \dfrac{(3 + \cos^2 2\theta)\cos 2\theta}{4}$.

2. If $\sin 5A - \sin 3A = \tan 2A - \tan A$,
then $\sin 8A = 4\sin A$.

3. Solve the equations

(1) $\sin kx = \sin lx$,

(2) $\cos ky = \sin ly$,

(3) $\tan kz = \cot lz$.

4. Find the values of θ and ϕ which satisfy the equations

$$2\sin(\theta-\phi)=1,$$
$$\sin(\theta+\phi)=1.$$

5. Shew that, if $\alpha+\beta+\gamma=2\pi$,

(1) $\sin^2\alpha=\sin^2\beta+\sin^2\gamma+2\sin\beta\sin\gamma\cos\alpha$,

(2) $\cot\alpha+\cot\beta+\cot\gamma$
$$=\cot\alpha\cot\beta\cot\gamma-\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma.$$

6. Find the angles of the triangle in which the sides are in the ratio of $1 : \sqrt{3} : 2$.

EXERCISE XXII.

1. Prove that in any triangle

$$^*2R\sin C=b\cos A+\sqrt{a^2-b^2\sin^2 A}.$$

2. Prove that

$$\frac{\sin A+\sin 3A+\sin 5A+\sin 7A+\sin 9A}{\cos A+\cos 3A+\cos 5A+\cos 7A+\cos 9A}$$
$$+\frac{\sin A+\sin 2A+\sin 3A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 3A+\cos 4A+\cos 5A}$$
$$=\frac{\sin 8A}{\cos 3A\cos 5A}.$$

3. If $\tan(\alpha+\beta)=\sqrt{3}$, and $\tan(\alpha-\beta)=\frac{1}{\sqrt{3}}$, find α and β.

4. Solve the equation

$$\cos 3\phi\sin^3\phi+\sin 3\phi\cos^3\phi=0.$$

* See page 8.

5. Shew that

(1) $$\frac{\cos\beta\cos\gamma\sin 2\alpha}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\frac{\cos\gamma\cos\alpha\sin 2\beta}{\sin(\beta-\gamma)\sin(\beta-\alpha)}+\frac{\cos\alpha\cos\beta\sin 2\gamma}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}=0,$$

(2) $$\frac{\sin\beta\sin\gamma\sin 2\alpha}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\frac{\sin\gamma\sin\alpha\sin 2\beta}{\sin(\beta-\gamma)\sin(\beta-\alpha)}+\frac{\sin\alpha\sin\beta\sin 2\gamma}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}=0,$$

(3) $$\frac{\cos(\beta\pm\gamma)\sin 2\alpha}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\frac{\cos(\gamma\pm\alpha)\sin 2\beta}{\sin(\beta-\gamma)\sin(\beta-\alpha)}+\frac{\cos(\alpha\pm\beta)\sin 2\gamma}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}=0.$$

EXERCISE XXIII.

1. Prove that, in any triangle,

$$\sin A+\sin B+\sin C=\frac{s}{R}.$$

2. Solve the equations

(1) $\tan 2x\cot x=\dfrac{2}{\sqrt{3}}+1.$

(2) $\cos x+\sin x=2\sqrt{2}\sin x\cos x.$

3. If the angle of a sector be 60°, compare the area of the circle with that of a circle inscribed in the sector.

4. Prove that

(1) $\tan^4\alpha+\cot^4\alpha=16\operatorname{cosec}^2 2\alpha\cot^2 2\alpha+2,$

(2) $\tan^4\alpha-\cot^4\alpha=8\operatorname{cosec}^4 2\alpha(\sin^2 2\alpha-2)\cos 2\alpha.$

5. Given
$$\log 18 = 1{\cdot}2552725$$
$$\log 24 = 1{\cdot}3802112,$$
find the logarithms of 405, ·1125, and $\dfrac{648}{(\cdot 027)^{\frac{1}{4}}}$.

6. Eliminate θ between
$$x = \cos\theta + \cos\frac{\theta}{2},$$
$$y = \sin\theta + \sin\frac{\theta}{2}.$$

EXERCISE XXIV.

1. Shew that in any triangle
$$c\cos B - b\cos C = \frac{c^2 - b^2}{a}.$$

2. Solve the equations

(1) $3\sin 2\theta = 3 + \cos 2\theta$.

(2) $4\cos 2\theta = 5 - 6\sin\theta$.

3. Shew that, if $\alpha + \beta + \gamma + \delta = \pi$, the sum of the products of $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and $\cos\delta$, taken two together, is equal to the sum of the products of $\sin\alpha$, $\sin\beta$, $\sin\gamma$, and $\sin\delta$, taken two together.

4. Prove that, in a triangle, if $A = 45°$ and $B = 60°$, then $2c = a(1 + \sqrt{3})$.

5. Simplify
$$\frac{\sin(\beta-\delta)\sin(\delta-\gamma)}{\cos\beta\cos\gamma}\sin 2\delta + \frac{\sin(\delta-\gamma)\cos\delta}{\sin(\beta-\gamma)\cos\beta}\sin 2(\delta-\beta)$$
$$+ \frac{\cos\delta\sin(\delta-\beta)}{\cos\gamma\sin(\gamma-\beta)}\sin 2(\delta-\gamma).$$

EXERCISE XXV.

1. If, in a triangle,

$$a^2 = b^2 - bc + c^2,$$

find A.

2. Solve the equations

(1) $\sqrt{2 \operatorname{vers} x - \operatorname{vers}^2 x} = \frac{1}{2} \operatorname{cosec} x$,

(2) $\cos 8x + 1 + \frac{1}{\sqrt{2}} \sin 8x \operatorname{cosec} 2x = \sin 8x$.

3. Eliminate θ between the equations

$$\tan\theta + \sin\theta = m,$$

$$\tan\theta - \sin\theta = n.$$

4. Prove geometrically that

(1) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.

(2) $\cos 2A = 2\cos^2 A - 1$.

5. Prove that

(1) $\cos(\delta - \alpha)\sin(\beta - \gamma) + \cos(\delta - \beta)\sin(\gamma - \alpha) = \cos(\delta - \gamma)\sin(\beta - \alpha)$.

(2) $\cos(\alpha + \beta)\cos(\alpha - \beta) + \sin(\beta + \gamma)\sin(\beta - \gamma) = \cos(\alpha + \gamma)\cos(\alpha - \gamma)$.

6. A man walking along a straight road observes that there is a conspicuous house in a direction making an angle 45° with the road. After going two miles further, he observes that the angle is 120°. Find the distance of the house from the road.

EXERCISE XXVI.

1. Prove that in any triangle

$$b\cos(D-C)+c\cos(D+B)=b\cos(D+C)+c\cos(D-B)$$
$$=a\cos D,$$

where D is any angle.

2. If in a triangle $a=3$, $b=4$, $C=60°$, find c to three places of decimals.

3. If $\quad \sin A=\sin B=\sin C=\sin D,$

and $\quad \sin(A+B)=\sin(C+D),$

then will $\quad \sin(A-C)=\sin(D-B).$

4. Solve the equations

(1) $\tan^2\theta+\cot^2\theta=14.$

(2) $\sqrt{3}\sin\theta+\sqrt{2}=\cos\theta.$

5. Shew that, in any triangle,

$$(a+b)\sin B=2b\sin\left(B+\frac{C}{2}\right)\cos\frac{C}{2}.$$

6. Calculate the value of e to six decimal places, and prove that $\log_a r=\log_a b\log_b c\log_c d\ldots\ldots\log_q r.$

EXERCISE XXVII.

1. Solve a triangle, given

$$\sin A=\tfrac{1}{2},\ \sin B=\tfrac{3}{5},\ a=10.$$

2. Calculate the lengths of the bisectors of the angles of the triangle whose sides are 12, 15, and 18 inches.

3. Shew that $\frac{bc}{4R^2} = \cos A + \cos B \cos C$

$$= \frac{*2\Delta}{a^2} \sin A.$$

4. Shew that the expression

$$2 \cos \frac{\alpha - \beta}{2} \cos \frac{\beta - \gamma}{2} \cos \frac{\gamma - \alpha}{2} + \sin^2 \frac{\beta - \gamma}{2} + \sin^2 \frac{\gamma - \alpha}{2} + \sin^2 \frac{\alpha - \beta}{2}$$

is constant for all values of α, β, γ.

5. Solve the equation

$$16 (\sin^4 \theta + 4 \cos^4 \theta) = 13.$$

6. C and D are points on a circle of which AB is a diameter. Shew that if C and D are on opposite sides of AB, then

$$AC.BD + AD.BC = AB.CD,$$

and that if they are on the same side, then

$$AC.BD + AB.CD = AD.BC.$$

EXERCISE XXVIII.

1. If in a triangle $2a = b + c$ then $\cot \frac{B}{2} \cot \frac{C}{2} = 3$.

2. Given that

$$\tan \alpha + \tan \beta + \tan \gamma + \tan \alpha \tan \beta \tan \gamma = 0,$$

prove that

$$\sin (\alpha + \beta + \gamma) + \sin (-\alpha + \beta + \gamma) + \sin (\alpha - \beta + \gamma) + \sin (\alpha + \beta - \gamma) = 0.$$

* See page 8.

3. Shew that, in any triangle

$$\frac{1}{4} - \frac{\Delta^2}{a^2 c^2}, \qquad \frac{1}{4} \cos A - \frac{\Delta^2}{a^2 bc}, \qquad \text{and} \quad \frac{1}{4} - \frac{\Delta^2}{a^2 b^2},$$

are in Geometrical Progression.

4. If $\quad x \cos \alpha - y \cos (\theta - \alpha) = y \sin \alpha + x \sin (\theta - \alpha),$
then $\quad y \sin (\theta - \alpha) + x \sin \alpha = x \cos (\theta - \alpha) - y \cos \alpha.$

5. On the sides of an equilateral triangle squares are described. Compare the area of the triangle formed by joining the centres of these squares, with that of the original triangle.

EXERCISE XXIX.

1. Shew that the smallest positive value of θ which satisfies the equation

$$125 \cos 3\theta - 450 \cos 2\theta + 915 \cos \theta - 562 = 0$$

is the least angle in a triangle whose sides are as $3 : 4 : 5$.

2. Shew that

$$\log_e \sqrt{\frac{(1+m)^{1-m}}{(1-m)^{1+m}}} = m + \frac{5}{2 \cdot 3} m^3 + \frac{9}{4 \cdot 5} m^5 + \frac{13}{6 \cdot 7} m^7 + \ldots$$

3. The cosines of two of the angles of a triangle are $\frac{1}{2}$ and $\frac{3}{5}$, and the side opposite to the first angle is $5\sqrt{3}$ inches; find the other sides.

4. Solve the equations

(1) $\sin^2 \theta + 2 \cos \alpha \cos \beta \cos \theta = \cos^2 \alpha + \cos^2 \beta.$

(2) $\sin 3\theta + \cos 2\theta \sin \theta = 2 \cos^3 \theta$

5. A tower is situated on a horizontal plane at a distance a from the base of a hill whose inclination is α. A person on the hill, looking over the tower, can just see a pond, the distance of which from the tower is b. Shew that if the distance of the observer from the foot of the hill be c the height of the tower is $\dfrac{bc\sin\alpha}{a+b+c\cos\alpha}$.

EXERCISE XXX.

1. If $(a^2+b^2)\sin(A-B)=(a^2-b^2)\sin(A+B)$, shew that the triangle is either isosceles or right-angled.

2. The sides of a triangle are in Arithmetical Progression. The greatest angle exceeds the least by 90°. Shew that the sides are in the ratio $\sqrt{7}+1 : \sqrt{7} : \sqrt{7}-1$.

3. If $\sin\alpha+\sin\theta=\cos\alpha+\cos\theta$, prove that the values of θ may be expressed by two infinite Arithmetical Progressions, the common differences of which are each equal to 2π.

4. If $xy+yz+zx=1$, prove by Trigonometry that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

5. If $$\frac{\sin\alpha}{\sin(\beta+\gamma)}=\frac{\sin(\alpha+\beta)}{\sin\gamma},$$

shew that $$\alpha+\beta+\gamma=m\pi \text{ or } \beta=n\pi.$$

6. In any triangle

$$a^2+b^2\cos^2 C=2ab\cos C+c^2\cos^2 B.$$

Properties of Triangles.

The fundamental formulæ of this portion of Trigonometry are

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c},$$

$$a^2=b^2+c^2-2bc\cos A,$$

$$R=\frac{a}{2\sin A}=\frac{abc}{4\Delta},$$

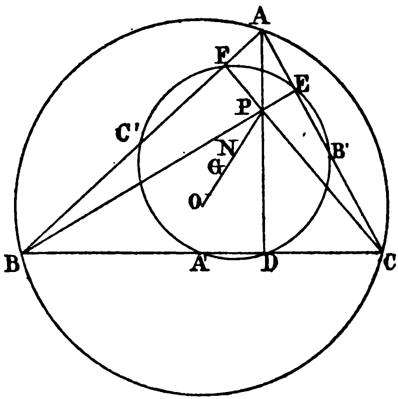
$$r=\frac{\Delta}{s}=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}},$$

$$r_a=\frac{\Delta}{s-a}=\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}.$$

The following theorems will be of great use to the student, whether in Trigonometry or in Pure Geometry.

Definition. The Nine Points Circle is the circle which passes through the feet of the perpendiculars.

In the appendix to Todhunter's *Euclid* it is proved that this circle also passes through the points of bisection of the sides, that its radius is $\frac{R}{2}$ and its centre N bisects OP, where O is the centre of the circumscribing circle, and P is the orthocentre. The Nine Points Circle also touches the inscribed and escribed circles of the triangle. The ordinary proofs of this theorem are very laborious, but a



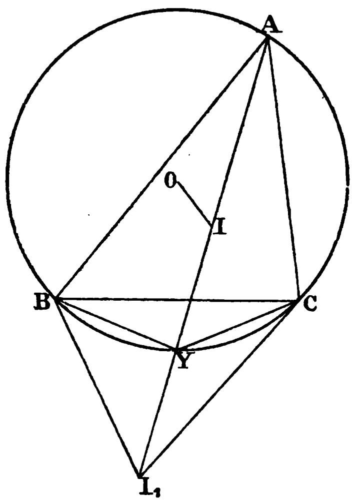
very elegant proof has been given in the Quarterly Journal of Mathematics by Mr Taylor of Clare College, which we recommend as by far the simplest we know. The proof is not introduced here, as it is founded on the method of inversion.

If G is the intersection of bisectors of sides, it may be proved by similar triangles that OG passes through P, and that $OG = \frac{1}{2} GP$. Hence we learn that O, G, N, P lie on a straight line. Since $PO = 2PN$, *i. e.* $PO : PN$ in the ratio of the radii, P is a centre of similitude with respect to the two circles. Therefore *every* line drawn from P to the circumscribing circle will be cut by the Nine Points Circle in the ratio of the radii; *i.e.* it will be bisected at that point.

Since $GO = 2GN$, G is the other centre of similitude.

Let us consider the triangle $I_1I_2I_3$ formed by the centres of the escribed circles. If I is the centre of the inscribed circle, we know that I_1I passes through A. Since AI_2 bisects the exterior angle at A, it is perpendicular to AII_1 which bisects A internally. Similarly AI_3

is perpendicular to AII_1; therefore I_2AI_3 is a straight line perpendicular to I_1IA. Thus we see that I is the orthocentre of the triangle $I_1I_2I_3$, and that the circle circumscribing ABC is the Nine Points Circle of $I_1I_2I_3$. The previous propositions may now be applied to the new system. For example: the radii of circles passing through any three of the points I, I_1, I_2, I_3, are all equal to $2R$.



Let AI meet the circumscribing circle in Y. Then II_1 is bisected in Y (since the Nine Points Circle bisects the line joining the orthocentre with each angular point of the triangle), and since IBI_1 is a right angle we have

$$YI = YI_1 = YB.$$

We can now find an expression for the distance OI.

We have $$AI \sin \frac{A}{2} = r,$$

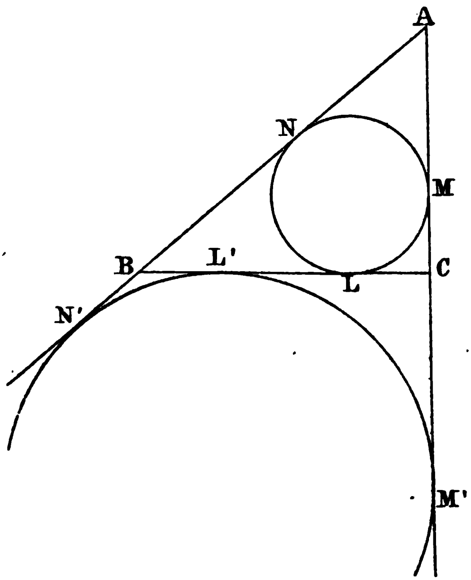
and $$YI = YB = 2R \sin \frac{A}{2},$$

$$\therefore AI \,.\, IY = 2Rr.$$

Now $$R^2 - OI^2 = \text{product of segments through } I$$
$$= AI \,.\, IY$$
$$= 2Rr;$$
$$\therefore OI^2 = R^2 - 2Rr.$$

Intercepts cut off by the Inscribed and Escribed Circles.

To express in terms of the sides the distances of the points of contact of the inscribed and escribed circles from the angular points of the triangle.



Let the inscribed circle touch at L, M and N. Then $CL = CM$, since both are tangents to the same circle.

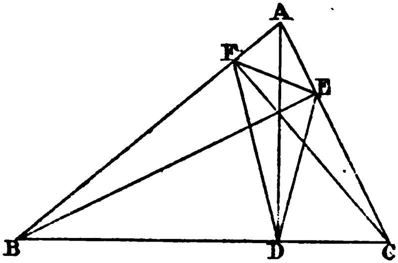
Thus $$2 \,.\, CL + 2AN + 2BN = \text{sum of sides},$$
or $$2CL + 2c = 2s;$$
$$\therefore CM = CL$$
$$= s - c.$$
Similarly $$AM = AN$$
$$= s - a,$$
and $$BL = BN$$
$$= s - b.$$

Let the escribed circle corresponding to A touch at L', M' and N'.

Then $$2AN' = AM' + AN'$$
$$= AB + BL' + AC + CL'$$
$$= 2s;$$
$$\therefore AM' = AN'$$
$$= s.$$
Thus $$BL' = BN'$$
$$= s - c$$
$$= CL,$$
and $$CL' = CM'$$
$$= s - b$$
$$= BL.$$

The Pedal Triangle.

The triangle DEF may be called the pedal triangle of ABC. Since $AF = b \cos A$ and $AE = c \cos A$, we see that, in the triangle FAE, the sides containing the angle A



are proportional to b and c. Therefore the triangles FAE and BAC are similar, and $EF = a\cos A$; in like manner $FD = b\cos B$ and $DE = c\cos C$. Also the angle $FEA = B$; similarly $DEC = B$: therefore $FED = \pi - 2B$. Thus the angles of the pedal triangle are $\pi - 2A$, $\pi - 2B$, and $\pi - 2C$.

Since, corresponding to every triangle, there is a triangle related in this way, we have the important theorem that *every formula connecting the sides and angles of a triangle remains true if for the sides we write* $a\cos A$, $b\cos B$, $c\cos C$, *and for the angles* $\pi - 2A, \pi - 2B, \pi - 2C$.

As an instance of this we will take the elementary relation

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

This becomes on substitution

$$a^2\cos^2 A = b^2\cos^2 B + c^2\cos^2 C + 2bc\cos B\cos C\cos 2A,$$

which is the same relation amongst the parts of the pedal triangle, expressed in terms of the sides and angles of the original triangle.

Formulæ may *sometimes* be simplified by the reverse process. To do this, we must write for the sides $a\operatorname{cosec}\dfrac{A}{2}$,

$b \operatorname{cosec} \frac{B}{2}$, $c \operatorname{cosec} \frac{C}{2}$, and for the angles $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}, \frac{\pi}{2} - \frac{C}{2}$. These values are obtained thus. The relations between the two triangles are :—

$$a \cos A = a',$$

$$\pi - 2A = A'.$$

Therefore we have
$$A = \frac{\pi}{2} - \frac{A'}{2},$$
$$a = a' \sec A$$
$$= a' \operatorname{cosec} \frac{A'}{2}.$$

As an instance of the application of this method consider the identity,

$$b \cos B \cos 2B + c \cos C \cos 2C + a \cos A \cos 2(B - C) = 0.$$

Transforming this according to the rule, we obtain

$$b \cos B + c \cos C = a \cos (B - C),$$

which is easily established. When $a \cos A$, $b \cos B$, $c \cos C$, appear in the identity as distinct quantities, (*i.e.* if, when a appears, it is always multiplied by $\cos A$, &c.), this method is usually applicable.

Exponential and Logarithmic Series.

In order to avoid the consideration of the convergency of the resulting series when a is greater than 2, it is better in establishing the series for a^x, to use the proof based on the expansion of $\left(1 + \frac{1}{n}\right)^{nx}$, which is given in

Todhunter's *Algebra*. We shall then deduce the expansion of $\log(1+x)$ as follows:

By the exponential theorem we have

$$a^z = e^{z\log a} = 1 + z\log a + \frac{z^2\,\overline{\log a}|^2}{\underline{|2}} + \&c.,$$

$$\therefore\ \frac{a^z - 1}{z} = \log a + z\frac{\overline{\log a}|^2}{\underline{|2}} + \text{terms in } z^2, \&c.$$

Therefore the limit of $\dfrac{a^z-1}{z}$, when z diminishes indefinitely, is $\log a$, all the other terms vanishing since the series is convergent. Using the symbol $L_{z=0}$ to denote this limit, we have

$$\log a = L_{z=0}\frac{a^z-1}{z}.$$

And, writing $1+x$ instead of a,

$$\log(1+x) = L_{z=0}\frac{(1+x)^z-1}{z},$$

$$= L_z\ \frac{z\,.\,x + \dfrac{z(z-1)}{\underline{|2}}x^2 + \dfrac{z(z-1)(z-2)}{\underline{|3}}x^3 + \&c.}{z}$$

$$= L_{z=0}\left\{x + \frac{z-1}{\underline{|2}}x^2 + \frac{(z-1)(z-2)}{\underline{|3}}x^3 + \&c.\right\}.$$

Now put $z=0$ and we arrive at the required series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.$$

We may obtain many identities by putting the same expression into two different forms, and then expanding the logarithms of each by the formula just proved for $\log(1+x)$.

For instance, to prove that

$$\log \sec^2 \theta + \sin 2\theta - \tfrac{1}{2} \sin^2 2\theta + \tfrac{1}{3} \sin^3 \theta \ldots\ldots\ldots\ldots (1)$$

is identically equivalent to

$$2 (\tan \theta - \tfrac{1}{2} \tan^2 \theta + \tfrac{1}{3} \tan^3 \theta \ldots\ldots\ldots) \ldots\ldots\ldots\ldots\ldots (2),$$

we proceed thus:

(1) is the expansion of $\log \sec^2 \theta + \log (1 + \sin 2\theta)$,

(2) is the expansion of $2 \log (1 + \tan \theta)$.

Now $\log \sec^2 \theta + \log (1 + \sin 2\theta)$

$$= \log \left(\frac{\cos \theta + \sin \theta}{\cos \theta}\right)^2 = 2 \log (1 + \tan \theta);$$

$\therefore$ the series (1) and (2) are equivalent.

Note. The expansion of $\log (1 + x)$ is not convergent when x is greater than unity, and therefore (2) is not convergent if $\tan \theta > 1$. If θ lies between $-\frac{\pi}{4}$ and $+\frac{\pi}{4}$ (1) and (2) are certainly equivalent.

EXERCISE XXXI.

1. Given that

$$\sin^3\theta \cos^5\theta = A \sin 8\theta + B \sin 6\theta + C \sin 4\theta + D \sin 2\theta,$$

find the values of A, B, C, and D.

2. Prove that $a^2 - b^2 = 2Rc \sin (A - B)$.

3. If

$$\sqrt{2} \cos \theta = \cos \alpha + \cos \beta,$$
$$\sqrt{2} \sin \theta = \cos \alpha - \cos \beta,$$
$$\sqrt{2} \cos \beta = \cos \theta + \cos \alpha,$$

find the numerical value of $\sin \theta$.

4. If the sides of a triangle are in Arithmetical Progression, prove that $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are also in Arithmetical Progression.

5. If $$\frac{\tan(\alpha+\beta-\gamma)}{\tan(\alpha-\beta+\gamma)}=\frac{\tan\gamma}{\tan\beta},$$
and β is *not* equal to γ,
then $$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0.$$

6. Solve the equations
$$\cos(2x+y)=\sin(x-2y),$$
$$\cos(x+2y)=\sin(2x-y).$$

EXERCISE XXXII.

1. Prove that, in any triangle,
$$a\cos A+b\cos B\cos 2C+c\cos C\cos 2B=0.$$

2. Given that $A=18^\circ$, $a=4$, and that the perpendicular from B on $AC=\sqrt{5}-1$, solve the triangle.

3. Eliminate θ from
$$\sin\theta(1-\cos\theta)=m,$$
$$\cos\theta(1-\sin\theta)=n.$$

4. Shew that
$$\frac{r}{R}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

5. Prove that
$$\log_e(x+1)=\log_e x+2\left\{\frac{1}{2x+1}+\frac{1}{3}\frac{1}{(2x+1)^3}+\dots\right\},$$

and thence calculate $\log_e 13$ to seven places of decimals, having given

$$\log_e 2 = \cdot 69314718,$$
$$\log_e 3 = 1\cdot 09861229.$$

EXERCISE XXXIII.

1. Prove that

$$\frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}} = \frac{c}{a-b} \text{ in any triangle.}$$

2. Shew that $\cot\theta + \cot 3\theta + \cot 5\theta = 0$ is satisfied by $\theta = n\frac{\pi}{3} + \frac{\pi}{6}$, where n is any integer.

3. *Prove that $\Delta^2 = rr_a r_b r_c$.

4. Shew that

$$\tan^{-1}\tfrac{2}{11} + 2\tan^{-1}\tfrac{1}{7} = \tan^{-1}\tfrac{1}{2}.$$

5. If $\dfrac{\sin(x-\alpha)}{\sin(\beta+\gamma)} = \dfrac{\cos x \cos\alpha}{\cos\beta\cos\gamma}$,

prove that

$$\frac{\sin(x-\beta)}{\sin(\gamma+\alpha)} = \frac{\cos x\cos\beta}{\cos\gamma\cos\alpha}.$$

6. Two sides of a triangle and the contained angle being given, if there be a small error in the measurement of the angle, investigate the consequent error in the calculated length of the third side.

* See page 8.

EXERCISE XXXIV.

1. Form the equation whose roots are $\cos\frac{\pi}{5}$ and $\cos\frac{2\pi}{5}$.

2. The angles of a triangle are as 3 : 5 : 7 and the radius of the circumscribing circle is 749·28. Calculate the lengths of the sides.

3. If
$$\cos\theta = \frac{\cos\phi - c}{1 - c\cos\phi},$$
then
$$c = \frac{\tan^2\frac{\theta}{2} - \tan^2\frac{\phi}{2}}{\tan^2\frac{\theta}{2} + \tan^2\frac{\phi}{2}}.$$

4. Shew that in any triangle
$$c^2\cos A\cos B + bc\cos A\cos C + ac\cos B\cos C = 2\Delta\sin C.$$

5. If $(a^2 + b^2)\cos 2A = b^2 - a^2$,

shew that either the sum or the difference of A and B is a right angle.

6. If $\alpha + \beta + \gamma = \pi$, prove that
$$\cot\frac{21\pi + \alpha}{2^7} + \cot\frac{21\pi + \beta}{2^7} + \cot\frac{21\pi + \gamma}{2^7}$$
$$= \cot\frac{21\pi + \alpha}{2^7}\cot\frac{21\pi + \beta}{2^7}\cot\frac{21\pi + \gamma}{2^7}.$$

EXERCISE XXXV.

1. If a is an arithmetical mean between b and c, then
$$b\cos C - c\cos B = 2(b - c).$$

2. Given a, b, and A; let c_1 and c_2 be the third sides. Shew that the distance between the centres of the circumscribing circles of the corresponding triangles is

$$\frac{c_1 \sim c_2}{2 \sin A}.$$

3. Solve generally the equations:

$$\sin \theta = \sin (\phi + \alpha),$$
$$\cos \theta = \cos (\phi + \beta).$$

4. Solve the equations:

(1) $\cos 4x + \sin 4x (1 + 2 \cos 2x - 2 \sin 2x) = 1$,

(2) $\cos^2 x \cos^2 \alpha = \cos^2 \alpha \sin^2 \alpha - 4 \cos x \sin (x - \alpha) \sin^3 \alpha$.

5. Shew that the product of the sines of the angles subtended by the sides at the centre of the inscribed circle, is one-fourth the sum of the sines of the angles of the triangle.

6. If $$\cos \frac{A}{2} = \frac{1}{2}\sqrt{\frac{b}{c} + \frac{c}{b}},$$

shew that the square described with one side of the triangle as diagonal is equal to the rectangle contained by the other two sides.

EXERCISE XXXVI.

1. In any triangle prove

$$A = \sin^{-1} \frac{a}{x}, \qquad B = \sin^{-1} \frac{b}{x}, \qquad C = \sin^{-1} \frac{c}{x},$$

and determine x.

2. Find B, given

$$a = 254{\cdot}13, \quad b = 375{\cdot}16, \quad A = 36^\circ\ 24',$$
$$\log 25413 = 4{\cdot}4050559,$$
$$\log 37516 = 4{\cdot}5742165.$$
$$L \sin 36^\circ\ 24' = 9{\cdot}7733614,$$
$$L \sin 61^\circ\ 10' = 9{\cdot}9425171,$$
$$L \sin 61^\circ\ 11' = 9{\cdot}9425866.$$

3. In any triangle

(1) $$\Delta = 4Rr \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

(2) $$2ab \cos A \cos B \cos 2A \cos 2B \cos 4C$$
$$= (c \cos C \cos 2C)^2 - (a \cos A \cos 2A)^2 - (b \cos B \cos 2B)^2.$$

4. If in a triangle

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = m^2,$$

then $$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1-m^2}{2}.$$

What are the limits of m?

5. If $x = \sin\alpha + \cos\alpha$, $y = \sin\beta + \cos\beta$,
shew that

$$xy = 2\sin\tfrac{1}{2}\left\{\frac{\pi}{2} + \sin^{-1}(x^2-1)\right\}\sin\tfrac{1}{2}\left\{\frac{\pi}{2} + \sin^{-1}(y^2-1)\right\}.$$

EXERCISE XXXVII.

1. If $$\frac{\sin(\theta-\phi)}{\sin\phi} = 1 + n,$$

and n is very small, prove that

$$\sin\phi = \left(1 - \frac{n}{2}\right)\sin\frac{\theta}{2} \quad \text{nearly.}$$

2. In a triangle C, c, and the sum m of the remaining sides are given. Shew that these sides are

$$m\cos^2\frac{\phi}{2} \text{ and } m\sin^2\frac{\phi}{2},$$

where $$m\sin\phi = \pm\sqrt{(m+c)(m-c)\sec^2\frac{C}{2}}.$$

3. If P is the orthocentre of a triangle and O the centre of the circumscribing circle, prove that

$$PO^2 = R^2\{3 + 2\cos 2A + 2\cos 2B + 2\cos 2C\}.$$

4. Shew that, if $\alpha+\beta+\gamma=\pi$,

$$\sin\alpha\sin(2\beta+\gamma)\sin(\beta+2\gamma)+\sin\beta\sin(2\gamma+\alpha)\sin(\gamma+2\alpha)$$
$$+\sin\gamma\sin(2\alpha+\beta)\sin(\alpha+2\beta)$$
$$=\sin 2\alpha\sin 2\beta\sin 2\gamma-\sin\alpha\sin\beta\sin\gamma.$$

5. Solve the equation

$$\frac{1}{2\sin 2\theta}=\frac{\cos\theta}{\cos\left(\frac{\pi}{6}+\theta\right)}-\frac{\sin\theta}{\sin\left(\frac{\pi}{6}+\theta\right)}.$$

6. If the straight line joining an angle of a triangle to the centre of the opposite escribed circle be bisected by the corresponding side, shew that the sides are in Arithmetical Progression.

EXERCISE XXXVIII.

1. Find the sine of half the supplement of $2\sin^{-1}\frac{3}{5}$.

2. The angular elevation of a column, as viewed from a station due North of it, being α, and as viewed from one

due East of the former station and at a distance c from it, being β, prove that the height of the tower is

$$\frac{c \sin \alpha \sin \beta}{\{\sin (\alpha + \beta) \sin (\alpha - \beta)\}^{\frac{1}{2}}}.$$

3. Shew that $\tan 20^\circ$, $\tan 80^\circ$, and $\tan 140^\circ$, are the roots of the equation

$$x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0.$$

4. If the tangents to the circumscribed circle at the vertices of the triangle ABC meet the opposite sides produced in D, E, F, respectively, prove that

$$\frac{1}{BE} = \frac{1}{AD} + \frac{1}{CF},$$

where B is neither the greatest nor the least angle of the triangle.

5. BC is a side of a square; on the perpendicular on BC through its middle point two points P, Q, are taken, equidistant from the centre of the square; BP, CQ, are joined, and intersect in A: prove that in the triangle ABC

$$\tan A (\tan B - \tan C)^2 + 8 = 0.$$

6. Eliminate θ from $\dfrac{\sin \theta}{\alpha} = \dfrac{\sin 3\theta}{\beta} = \dfrac{\sin 5\theta}{\gamma}$.

EXERCISE XXXIX.

1. Find B and C, given $A = 120^\circ$, $a = 8$, $b = 7$,

$$\log 7 = \cdot 8450980, \qquad \log 8 = \cdot 9030900,$$

$$L \sin 60^\circ = 9 \cdot 9375306,$$

$$L \sin 49^\circ 16' = 9 \cdot 8795287, \qquad L \sin 49^\circ 17' = 9 \cdot 8796375.$$

2. The sides of a quadrilateral inscribed in a circle are 240, 340, 396, and 424; calculate the radius of the circle.

3. The three escribed circles of a triangle touch the three sides in the points D, E, F, respectively. Prove that

$$\frac{EF^2}{BC}-BC=\frac{FD^2}{CA}-CA=\frac{DE^2}{AB}-AB.$$

4. Straight lines are drawn through the centres of the inscribed and escribed circles; if the products of the segments cut off these lines by the circumscribing circle are t^2, t_1^2, t_2^2, and t_3^2, respectively, prove that

$$\frac{1}{t^2}=\frac{1}{t_1^2}+\frac{1}{t_2^2}+\frac{1}{t_3^2}=\frac{a+b+c}{abc}.$$

5. If the circle circumscribing an isosceles triangle be equal to the escribed circle which touches one of the equal sides, prove that the triangle is right-angled.

6. Solve the equations

(1) $\cos\frac{\theta}{2}-\sin\frac{\theta}{2}+\cos\theta\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)=0.$

(2) $\cot\left(\phi+\frac{\pi}{6}\right)+\cot\left(\phi-\frac{\pi}{6}\right)+\cot\left(\phi+\frac{\pi}{2}\right)+\tan 3\phi=0.$

EXERCISE XL.

1. In any triangle

$$a\cos A\cos 2A+b\cos B\cos 2B\cos 4C$$
$$+c\cos C\cos 2C\cos 4B=0.$$

2. In any triangle

$$R=\frac{a\cos A+b\cos B+c\cos C}{4\sin A\sin B\sin C}.$$

3. If $\tan\theta$ be less than unity, shew that

$$2\sin^2\theta+\tfrac{1}{2}\,.\,4\sin^4\theta+\tfrac{1}{3}\,.\,8\sin^8\theta+\dots\ ad\ inf.$$
$$=2\,\{\tan^2\theta+\tfrac{1}{3}\tan^6\theta+\tfrac{1}{5}\tan^{10}\theta+\dots\ ad\ inf.\}$$

4. If A, B, and C, are the angles of a triangle

$$\cos^3 A+\cos^3 B+\cos^3 C$$
$$=3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}-\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}+1.$$

5. The sides of a triangle are in Arithmetical Progression; shew that the product of the greatest and least sides is equal to six times the product of the radii of the inscribed and circumscribed circles.

6. A person notices two objects in a straight line, in a direction due East; after walking a distance a due North, he observes that the objects subtend an angle α at his eye; after a further distance a, an angle β is subtended: prove that the distance between the objects is

$$\frac{3\tan\alpha\tan\beta}{2\tan\alpha-\tan\beta}\,a.$$

EXERCISE XLI.

1. Shew that $\tan^{-1}\frac{5}{6}+\tan^{-1}\frac{1}{11}=45^\circ.$

2. Find the greatest value of

$$\frac{\operatorname{cosec}^2\theta-\tan^2\theta}{\cot^2\theta+\tan^2\theta-1}.$$

3. If A, B, C, are the angles of a triangle, prove that

$$\tan B\tan C+\tan C\tan A+\tan A\tan B-1$$
$$=\sec A\sec B\sec C.$$

4. In an acute-angled triangle ABC another triangle is inscribed; shew that the least value of the perimeter of the second triangle is

$$a\cos A + b\cos B + c\cos C.$$

5. If the sides of a triangle are 3108, 2331, 3885, find the radii of the inscribed and circumscribed circles, and the distance between their centres.

6. If N is the centre of the Nine Points Circle of a triangle, and A', B', C' are the middle points of the sides, prove that

$$BC\cos NA'C + CA\cos NB'A + AB\cos NC'B = 0.$$

7. Prove that the radius of the inscribed circle of the pedal triangle $= 2R\cos A\cos B\cos C$.

EXERCISE XLII.

1. Prove that $2\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{7} = \dfrac{\pi}{4}$.

2. The sides of a given triangle are 11, 23, 29 and its area is Δ; find a similar triangle whose area $= 35721\Delta$.

3. If A, B, C, and A', B', C', be the angles of two triangles such that

$$A - A' = \frac{B' - B}{2} = C - C',$$

and $$\cos A\cos B\cos C = \cos A'\cos B'\cos C',$$

then will $$\cos B' = 2\cos C\cos A.$$

4. If N is the centre of the Nine Points Circle, prove that

$$AN = \frac{R}{2}\sqrt{1 + 8\cos A \sin B \sin C}.$$

5. I observe the angular elevation of the summits of two spires (which appear in a straight line) to be α, and the angular depressions of their reflexions in still water to be β and γ. If the height of my eye above the level of the water is c, then the horizontal distance between the spires is

$$\frac{2c\cos^2\alpha \sin(\beta-\gamma)}{\sin(\beta-\alpha)\sin(\gamma-\alpha)}.$$

6. Find (by means of a table of logarithms) the value of $\dfrac{(\cdot 0567)^{\frac{2}{5}} \times (34\cdot 62)^{\frac{1}{6}}}{(2\cdot 396)^{\frac{7}{5}}}$ to five places of decimals.

EXERCISE XLIII.

1. Shew that

(1) $\tan(A+60^\circ)\tan(A-60^\circ) + \tan A \tan(A+60^\circ) + \tan(A-60^\circ)\tan A = -3.$

(2) $\cot(A+60^\circ)\cot(A-60^\circ) + \cot A \cot(A+60^\circ) + \cot A \cot(A-60^\circ) = -3.$

2. Solve the equations:

(1) $\cos^{-1}\dfrac{\sqrt{3}}{2} + \sec^{-1}x = \dfrac{\pi}{4}.$

(2) $\sin^{-1}x = \tan^{-1}2x.$

(3) $\sin^{-1}\sqrt{x} + \sin^{-1}\sqrt{2x} = \dfrac{5\pi}{6}.$

3. Shew that

$$\sec\frac{\pi}{2^2}\sec\frac{\pi}{2^3}\sec\frac{\pi}{2^4}\ldots\ldots=\frac{\pi}{2},$$

the number of factors being infinite.

4. AD is a diameter of the circle circumscribing the triangle ABC; the tangent at D meets AC produced in F. Shew that the area of the triangle CDF is $\frac{1}{2}b^2\cot^3 B$.

5. If α, β, γ, be the radii of three circles passing through the centre of the circumscribed circle and of which respectively the sides of the triangle are chords; prove that

$$\frac{1}{bc\alpha}+\frac{1}{ca\beta}+\frac{1}{ab\gamma}=\frac{1}{R^3}.$$

6. If N is the centre of the Nine Points Circle of the triangle ABC, and if the middle points of BC and CA are A' and B', shew that the area of the quadrilateral $NA'CB'$ is $\frac{\rho^2}{2}\{\sin 2A+\sin 2B+2\sin 2C\}$, where ρ is the radius of the Nine Points Circle.

EXERCISE XLIV.

1. Prove that

$$\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}.$$

2. Find the numerical distance between the orthocentre and the centre of the circumscribing circle in a triangle whose sides are 7, 8, 9.

3. Eliminate α from the equations:

$$\frac{\cos 3\alpha}{a}+\frac{\cos 2\alpha}{b}+\frac{\cos\alpha}{c}=0,$$

$$\frac{\cos 2\alpha}{a}+\frac{\cos\alpha}{b}+\frac{1}{2c}=0.$$

4. If the perpendiculars from the centre of the circumscribing circle of a triangle on the sides are α, β, γ, respectively; shew that

$$4\left(\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}\right)=\frac{abc}{\alpha\beta\gamma}.$$

5. Two men, of heights a and a', stand successively on the top of a column; the angles subtended by these two men at the eye of a stationary observer are α and α'; prove that, when the observer views the summit of the column, his line of sight is inclined to the column at an angle ϕ such that

$$\cot\phi=\frac{a\cot\alpha-a'\cot\alpha'}{a-a'}.$$

6. If $\sin^3\theta=\sin(\alpha-\theta)\sin(\beta-\theta)\sin(\gamma-\theta)$,

and $$\alpha+\beta+\gamma=\pi,$$

prove that $$\cot\theta=\cot\alpha+\cot\beta+\cot\gamma.$$

EXERCISE XLV.

1. Prove that

$$\tan^{-1}x+\tan^{-1}y+\tan^{-1}\frac{1-x-y-xy}{1+x+y-xy}=\frac{\pi}{4}.$$

2. Shew that in any triangle

$$a^2,\quad b^2-c^2,\ \text{and}\ b^2+c^2-2bc\cos(B-C)$$

are in Geometrical Progression.

3. Solve a triangle, given

$$R=2, \qquad \Delta=\sqrt{3}, \qquad r_b=1.$$

Also prove that in this triangle

(1) *$4rr_a=a^2$,

(2) $r_c=p_a=r_b$.

4. Assuming the radius of the earth to be 4000 miles, express in inches the difference between its circumference and the perimeter of a regular polygon of 10,000 sides which is inscribed in it.

5. If in a triangle,

$$^*r_a : r_b : r_c = 1 : 2 : 3,$$

find $\dfrac{R}{r}$.

6. Through A, one of the angles of a triangle ABC, a straight line QR is drawn parallel to BC, and through B and C perpendiculars QBP, RCP are drawn to BA and CA respectively. Shew that the area of the triangle

$$PQR=\frac{\{BC\cos(B-C)\}^2}{2\sin A\cos B\cos C}.$$

EXERCISE XLVI.

1. Shew that the equation $2\tan\theta=\cos\theta$ has a root between 18° and 30°.

2. Prove that

$$\Delta^2=\frac{abc}{8}(a\cos A+b\cos B+c\cos C).$$

* See page 8.

3. Shew that

$$2\sin^{-1}\tfrac{4}{5} = \sin^{-1}\tfrac{24}{25} = \cos^{-1}\tfrac{7}{25}.$$

4. Shew that, in any triangle,

$$\begin{aligned}&(\sin A + \sin B)(\cos B + \cos C)(\cos C + \cos A)\\ &\quad + (\sin B + \sin C)(\cos C + \cos A)(\cos A + \cos B)\\ &\quad + (\sin C + \sin A)(\cos A + \cos B)(\cos B + \cos C)\\ &\quad = (\sin A + \sin B)(\sin B + \sin C)(\sin C + \sin A).\end{aligned}$$

5. Find the values of the radii of the escribed circles of the pedal triangle of the triangle ABC.

6. ABC is a triangle; on BA is measured $BD = AC$; BC and AD are bisected in E and F; EF is joined; shew that the radius of the circle described around BEF is

$$\frac{BC}{4}\operatorname{cosec}\frac{A}{2}.$$

7. Eliminate ϕ from $x = 3\cos\phi + \cos 3\phi$,

$$y = 3\sin\phi - \sin 3\phi.$$

EXERCISE XLVII.

1. Shew that all the angles satisfying $\sin 2\theta = \frac{4}{5}$ are obtained by taking the angles $\tan^{-1} 2$ and their complements.

2. The sides of a triangle are $5n$, $2n^2 + 2$, and $2n^2 + 3n - 2$. Find r.

3. Shew that

(1) $\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{7}{9}$,

(2) $\tan^{-1}\dfrac{3}{4} = \dfrac{1}{2}\tan^{-1}\dfrac{24}{7} = \dfrac{1}{3}\tan^{-1}\left(-\dfrac{117}{44}\right)$.

4. If
$$\frac{\tan y}{\tan \beta} = \frac{\sin (x - \alpha)}{\sin \alpha}$$
and
$$\frac{\tan y}{\tan 2\beta} = \frac{\sin (x - 2\alpha)}{\sin 2\alpha},$$
then
$$\frac{\tan y}{\sin 2\beta} = \frac{\sin x}{\sin 2\alpha} = \frac{\cos x}{\cos 2\alpha - \cos 2\beta}.$$

5. Shew that, if $\alpha + \beta + \gamma = \pi$,
$$2\frac{\tan^3 \alpha + \tan^3 \beta + \tan^3 \gamma}{\tan \alpha + \tan \beta + \tan \gamma}$$
$$= 3(\tan^2 \alpha + \tan^2 \beta + \tan^2 \gamma) + 6 - \tan^2 \alpha \tan^2 \beta \tan^2 \gamma$$
$$= 2\tan^2 \alpha \tan^2 \beta \tan^2 \gamma + 6 \sec^2 \alpha (1 - \tan \beta \tan \gamma).$$

6. I is the centre of the circle inscribed in the triangle ABC. Prove that

(1) $IB \,.\, IC = \dfrac{\Delta}{s} a \sec \dfrac{A}{2}$,

(2) $IA \,.\, IB \,.\, IC = abc \dfrac{\Delta}{s^2}$.

EXERCISE XLVIII.

1. Prove that
$$\tan^{-1} \tfrac{1}{3} - \tan^{-1} \tfrac{1}{5} + \tan^{-1} \tfrac{1}{7} = \tan^{-1} \tfrac{8}{11}.$$

2. Shew that, in any triangle,
$$4 \sin^2 A \sin^2 B \sin^2 C + \sin^4 A + \sin^4 B + \sin^4 C$$
$$= 2(\sin^2 B \sin^2 C + \sin^2 C \sin^2 A + \sin^2 A \sin^2 B).$$

3. If $\alpha + \beta + \gamma = \dfrac{\pi}{4}$, prove that
$$\tan \alpha + \tan \beta + \tan \gamma + \tan \beta \tan \gamma + \tan \gamma \tan \alpha + \tan \alpha \tan \beta$$
$$= 1 + \tan \alpha \tan \beta \tan \gamma.$$

4. To solve a triangle let a, c, and C be given; let b_1, b_2 be the two values of the third side and r_1 r_2 those of the radii of the inscribed circles. Prove that

$$(1) \qquad \left(\frac{b_1}{r_1} - \cot\frac{C}{2}\right)\left(\frac{b_2}{r_2} - \cot\frac{C}{2}\right) = 1,$$

$$(2) \qquad r_1 r_2 = a(a-c)\sin^2\frac{C}{2}.$$

5. ABC and ABD are equilateral triangles having the side AB common; with A, B, and D as centres three circles are described, the radius of each being equal to AB. Prove that, if a circle be described touching the arcs AC, CB, BA, its radius will be equal to

$$\frac{3\sqrt{3}-1}{13}AB.$$

6. Prove that

$$\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \tfrac{3}{16}.$$

EXERCISE XLIX.

1. Prove that

$$\tan^{-1}\frac{m}{m+1} + \tan^{-1}\frac{1}{2m+1} = n\pi + \frac{\pi}{4}.$$

2. Prove that

$$\frac{a^2+b^2+c^2}{8R^2} = 1 + \cos A \cos B \cos C.$$

3. Prove that if the roots of a cubic equation are the tangents of the angles of a triangle, the equation has the form

$$x^3 + px^2 + qx + p = 0;$$

also, if the roots are the cotangents, the equation has the form

$$x^3 + px^2 + x + q = 0.$$

4. Shew that, if A, B, C are the angles of a triangle,

$$\tan\frac{31\pi+3B}{2^6}\tan\frac{31\pi+3C}{2^6}+\tan\frac{31\pi+3C}{2^6}\tan\frac{31\pi+3A}{2^6}$$
$$+\tan\frac{31\pi+3A}{2^6}\tan\frac{31\pi+3B}{2^6}=1.$$

5. $ABCD$ is a quadrilateral inscribed in a circle, whose diagonals intersect at right angles; express the lengths of the sides, the diagonals, and the segments of the diagonals, in terms of the diameter of the circle, and of the angles BAC (α) and DAC (β). Deduce the formulæ for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$.

6. Given $\log 2 = \cdot 30103$, determine x from the equation
$$8^x \times 125^{2-x} = 2^{4x+3} \times 5^x.$$

EXERCISE L.

1. The radii of two circles are connected by the relation
$$R : r :: \sqrt{3}+1 : \sqrt{3}-1,$$
and the distance between their centres is $2R-2r$; find the angles which they subtend at their centres of similitude.

2. If $\cos A$, $\cos B$, $\cos C$ are in Arithmetical Progression, prove that $\cot I_1$, $\cot I_2$, $\cot I_3$ are also in Arithmetical Progression, where I_1, I_2, I_3 are the centres of the escribed circles of the triangle ABC.

3. Solve a triangle, having given the radii of the inscribed and circumscribing circles, and the angle subtended at A by the distance between their centres.

4. A man at a distance c from a straight line of railway sees a train standing upon the line, which

has its nearer end at a distance a from the point in the railway nearest to him. He observes the angle α which the train subtends at his eye, and thence calculates its length. If, in observing the angle α, he makes a small error θ, prove that the corresponding error in the calculated length of the train has to its true length the ratio

$$\frac{c\,\theta}{\sin\alpha\,(c\cos\alpha - a\sin\alpha)}.$$

5. Prove that

$$\log_e\sqrt{y+1} - \log_e y + \log_e\sqrt{y-1}$$
$$= \frac{1}{1-2y^2} + \frac{1}{3}\left(\frac{1}{1-2y^2}\right)^3 + \frac{1}{5}\left(\frac{1}{1-2y^2}\right)^5 + \ldots\ ad\ inf.$$

and thence shew that

$$\log_{10}\frac{25}{24} = 2\mu\left\{\left(\frac{1}{7}\right)^2 + \frac{1}{3}\left(\frac{1}{7}\right)^6 + \frac{1}{5}\left(\frac{1}{7}\right)^{10} + \ldots\ ad\ inf.\right\},$$

where μ is the *modulus*.

6. Prove that in any triangle

$$a\cot A + b\cot B + c\cot C = 2R + 2r.$$

EXERCISE LI.

1. Solve by Trigonometry the equation

$$x^3 - 6x^2 + 9x - 3 = 0.$$

2. Shew that the sum of the five smallest angles whose tangents are respectively $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{36}{37}$, is π.

3. A circle is described on AB as diameter. From any point P on the circle PM is drawn perpendicular to AB. Prove that

$$2\frac{AM}{AB} + \frac{PM^2 - AM^2}{AP^2} = 1.$$

4. A tower has a mound of earth against its base, in the shape of a plane inclined at an angle δ to the horizon; also A is a point in the horizontal plane through the foot of the tower and mound. A man starts from A and walks directly towards the tower, first to C, the foot of the mound, and then up the mound to a point B. At A the elevation of the tower is α above the horizontal plane; at B it is β above the plane of the mound. Given that $AC = a$ and $BC = b$, prove that the height of the tower above AC is equal to

$$\sin \alpha \frac{a \sin (\beta + \delta) + b \sin \beta}{\sin (\beta + \delta - \alpha)}.$$

5. Solve the equation

$$\tan^{-1} (x - 2) + \tan^{-1} (x + 2) + \tan^{-1} (2x) = \tan^{-1} (4x).$$

6. If $$u = \sec \theta - \sec \phi,$$
and $$v = \tan \frac{\theta}{2} \cot \frac{\phi}{2},$$
prove that $\tan \theta \tan \phi = \dfrac{2uv}{v^2 - 1}$.

7. Find the area of the triangle whose sides are

$$ab (c^2 + d^2), \; cd (a^2 + b^2), \text{ and } (bc + ad)(ac - bd).$$

EXERCISE LII.

1. Given that $a^2 + b^2 = 1$, and that

$$\log 2 = \cdot 30103, \qquad \log (1 + a) = \cdot 1928998,$$
$$\log (1 + b) = \cdot 2622226,$$

find $\log (1 + a + b)$.

2. Prove that the equation $\tan x = kx$ has an infinite number of real roots.

3. If $\dfrac{\cos(\alpha+\beta+\theta)}{\sin(\alpha+\beta)\cos^2\gamma} = \dfrac{\cos(\gamma+\alpha+\theta)}{\sin(\gamma+\alpha)\cos^2\beta}$ where β and γ are unequal, prove that each member will be equal to $\dfrac{\cos(\beta+\gamma+\theta)}{\sin(\beta+\gamma)\cos^2\alpha}$, and that

$$\cot\theta = \frac{\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)}{\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)+\sin^2(\alpha+\beta+\gamma)}.$$

4. In any triangle, given that

$$x+\frac{1}{x}=2\cos A, \text{ and } y+\frac{1}{y}=2\cos B,$$

prove that $$bx+\frac{a}{y}=ay+\frac{b}{x}=c.$$

5. In a triangle ABC, $AB = AC + \frac{1}{2}BC$, and BC is divided in O so that $BO : OC :: 1 : 3$; prove that the angle ACO is twice the angle AOC.

6. Adapt to logarithmic computation

(1) $x=\sqrt{a+b}\pm\sqrt{a-b}$,

(2) $x=\sqrt{\dfrac{a+b}{a-b}}\pm\sqrt{\dfrac{a-b}{a+b}}$,

(3) $a\sin x + b\cos x = c$.

EXERCISE LIII.

1. If $A+B+C+D=2\pi$, then

$$\begin{aligned}\cos\tfrac{1}{2}A\cos\tfrac{1}{2}D\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C-\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\sin\tfrac{1}{2}A\sin\tfrac{1}{2}D\\ =\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A+C)\cos\tfrac{1}{2}(A+D)\\ =\tfrac{1}{4}\{\cos A-\cos B-\cos C+\cos D\}.\end{aligned}$$

2. A triangle has sides 15 feet and 18 feet, and the contained angle is 60° 0′ 10″. Calculating the third side for an angle of 60°, find the correction to be applied for the 10″.

3. Solve the equation

$$\sin^{-1} mx + \sin^{-1} nx = k\pi.$$

4. Equilateral triangles are described on the sides of a triangle outwards, and their centres are joined. Shew that the triangle thus formed is equilateral.

5. ABC is a triangle and I the centre of its inscribed circle. Shew that AI passes through the centre of the circle described round BIC.

6. If $\dfrac{\sin\psi}{\sin\phi} = \dfrac{3+\sin^2\phi}{1+3\sin^2\phi}$ express $\sin\phi$ in terms of ψ.

EXERCISE LIV.

1. Shew that

$$\tan 4\theta = \frac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta}.$$

2. Find the general value of

$$\tan^{-1}\{\sin(\cos^{-1}\sqrt{\tfrac{2}{3}})\}.$$

3. If A, B, and C are the angles of a triangle, prove that

$$\tan^4 A + \tan^4 B + \tan^4 C - 4\tan^2 A \tan^2 B \tan^2 C$$
$$= (\tan^2 A + \tan^2 B + \tan^2 C)^2 - 2(1 + \sec A \sec B \sec C)^2.$$

4. $A'B'C'$ is the pedal triangle of ABC. Prove that

$$R(R_a^2 + R_b^2 + R_c^2) + 2R_a R_b R_c = R^3,$$

where R, R_a, R_b, R_c are the radii of the circles either inscribed in, or described about, the triangles ABC, $AB'C'$, $A'BC'$, $A'B'C$.

5. A and B are two stations in the same vertical plane with O. The altitude (h) of A above the horizontal plane through O is known, and that of B (H) is to be found. The inclinations of AO, BO, AB to the vertical are observed to be α, β, γ respectively. Shew that

$$H = h\frac{\cos\beta\sin(\gamma-\alpha)}{\cos\alpha\sin(\gamma-\beta)}.$$

6. Given $$\sin\eta + \sin\zeta = 2p,$$
and $$\cos\eta + \cos\zeta = 2q,$$
express $\cos 2\eta + \cos 2\zeta$ in terms of p and q.

EXERCISE LV.

1. Prove that in any triangle

$$a^2+b^2+c^2 = (b^2+c^2)\cos^2 A + (c^2+a^2)\cos^2 B + (a^2+b^2)\cos^2 C$$
$$+ 2bc\cos B\cos C + 2ca\cos C\cos A + 2ab\cos A\cos B$$
$$= 2\{bc\cos A + ca\cos B + ab\cos C\}.$$

2. Extract the square root of

$$\frac{c^4}{4} + 2c^2\Delta\cot C - 4\Delta^2.$$

3. If the hypothenuse of a right-angled triangle be given, prove that, in order that the distance between the centres of the inscribed and circumscribed circles may be as small as possible, the triangle must be isosceles.

4. If, in the triangle ABC, the perpendiculars from B and C are BE and CF, and if EF and BC produced meet in Q, prove that

$$2(QE^2 - QF^2) = (BQ^2 - CQ^2)(\cos 2B + \cos 2C).$$

5. If I is the centre of the inscribed circle of the triangle ABC, prove that

$$a\,.\,IA^2 + b\,.\,IB^2 + c\,.\,IC^2 = abc.$$

6. Prove that in any triangle

$$*OP^2 = R^2\{1 - 8\cos A\cos B\cos C\}.$$

EXERCISE LVI.

1. If in a given triangle $a^4 + b^4 + c^4 = 2c^2(a^2 + b^2)$, find the angle C.

2. If $$A + B + C = 180^\circ,$$

and $$\alpha = \frac{1 + \tan\frac{A}{4}}{1 - \tan\frac{A}{4}}, \quad \beta = \frac{1 + \tan\frac{B}{4}}{1 - \tan\frac{B}{4}}, \quad \gamma = \frac{1 + \tan\frac{C}{4}}{1 - \tan\frac{C}{4}},$$

prove that $$\alpha + \beta + \gamma = \alpha\beta\gamma.$$

3. If a, b, c are the sides of a triangle, and

$$\cos\theta = \frac{a}{b + c}, \quad \cos\phi = \frac{b}{c + a}, \quad \cos\psi = \frac{c}{a + b},$$

prove that

$$\tan^2\frac{\theta}{2}\tan^2\frac{\phi}{2}\tan^2\frac{\psi}{2} = \tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2}.$$

4. If α, β, γ be any three angles and σ half their sum, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1$$
$$= 4\cos\sigma\cos(\sigma - \alpha)\cos(\sigma - \beta)\cos(\sigma - \gamma).$$

5. Solve the equation

$$2\tan\theta + \tan(\alpha - \theta) = \tan(\beta + \theta).$$

6. If a, b, c, d are the lengths of the sides of a quadrilateral, and $\widehat{bc}$ represents the angle between b and c, prove that

$$d^2 = a^2 + b^2 + c^2 - 2bc\cos\widehat{bc} - 2ca\cos\widehat{ca} - 2ab\cos\widehat{ab}.$$

* See figure, page 40.

EXERCISE LVII.

1. Given $a = 1000$, $b = 1982{\cdot}8$, $C = 67^\circ\, 30'\, 40''$, find $\dfrac{R}{r}$.

2. If $\alpha + \beta + \gamma = \dfrac{\pi}{4}$, prove that

$$(1 - \tan\alpha)(1 + \tan\beta)(1 + \tan\gamma)(\sin 2\beta - \sin 2\gamma)$$
$$+ (1 - \tan\beta)(1 + \tan\gamma)(1 + \tan\alpha)(\sin 2\gamma - \sin 2\alpha)$$
$$+ (1 - \tan\gamma)(1 + \tan\alpha)(1 + \tan\beta)(\sin 2\alpha - \sin 2\beta) = 0.$$

3. Prove that in any triangle

$$b^2 \cos^2 B \sin 4C + 2bc \cos B \cos C \sin 2(C - B) = c^2 \cos^2 C \sin 4B.$$

4. Eliminate θ and ϕ from the equations

$$\frac{a\cos\theta\sec\phi - x}{a\sin(\theta+\phi)} = \frac{y - b\sin\theta\sec\phi}{b\cos(\theta+\phi)} = \tan\phi.$$

5. If α, β, γ be the sides of the triangle formed by joining the centres of the escribed circles, and $2S = A + B + C$, prove that

(1) $$\beta^2 \sin^2\frac{B}{2} - \gamma^2 \sin^2\frac{C}{2} = \alpha \sin\frac{A}{2}\left(\beta\sin\frac{B}{2}\cos C - \gamma\sin\frac{C}{2}\cos B\right).$$

(2) $$\sqrt{2}\,\gamma = (\alpha - \beta)\frac{\cos\frac{C}{4} + \sin\frac{C}{4}}{\sin\frac{B-A}{4}} = (\alpha+\beta)\frac{\cos\frac{C}{4} - \sin\frac{C}{4}}{\cos\frac{B-A}{4}}.$$

(3) $$(\alpha - \beta)\left\{\cos\frac{S-A}{2} + \cos\frac{S-B}{2} + \sin\frac{S-A}{2} + \sin\frac{S-B}{2}\right\}$$
$$= (\alpha+\beta)\left\{\cos\frac{S-B}{2} - \sin\frac{S-B}{2} - \cos\frac{S-A}{2} + \sin\frac{S-A}{2}\right\}.$$

EXERCISE LVIII.

1. If $\tan A = \frac{1}{3}$, $\tan B = \frac{1}{5}$, $\tan C = \frac{1}{7}$, $\tan D = \frac{1}{8}$, find the value of $A + B + C + D$.

2. If $$\alpha + \beta + \gamma = \frac{\pi}{2},$$
prove that
$$(\cos\alpha + \cos\beta + \cos\gamma)(\cos\beta + \cos\gamma - \cos\alpha)$$
$$\times (\cos\gamma + \cos\alpha - \cos\beta)(\cos\alpha + \cos\beta - \cos\gamma)$$
$$= 4\cos^2\alpha\cos^2\beta\cos^2\gamma.$$

3. Defining the hyperbolic sine and cosine of x (abbreviated as $\sinh x$ and $\cosh x$) as follows:
$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}),$$
establish the relations $\cosh^2 x - \sinh^2 x = 1$
$$\cosh 2x = 1 + 2\sinh^2 x$$
$$= 2\cosh^2 x - 1.$$

Express $\tanh 2x$ in terms of $\tanh x$.

4. Determine ϕ from the equation $\tan^3\phi = \tan(\phi + \alpha)$.

5. Find the area of the triangle formed by those tangents of the three escribed circles which are parallel to the sides of the triangle and touch them externally.

6. A quadrilateral, whose sides, taken in order, are in G. P., the common ratio being r, is inscribed in a circle. Shew that
$$\frac{\tan ABC}{\tan BCD} = \frac{r^2 - 1}{r^2 + 1}.$$

EXERCISE LIX.

1. Find the tangent of half the complement of $2\tan^{-1}\frac{1}{11}$.

2. Solve by Trigonometry

$$x^3 - 3x^2 - 6x + 17 = 0.$$

3. In a triangle $c = 16$, $a + b = 20$, and the distance of C from the middle point of $AB = 9$. Find a and b.

4. Prove that

$$\{7\log\tfrac{16}{15} + 5\log\tfrac{25}{24} + 3\log\tfrac{81}{80}\}\{16\log\tfrac{16}{15} + 12\log\tfrac{25}{24} + 7\log\tfrac{81}{80}\} = \log\{5^{\log 2}\}.$$

5. Let α, α_1, α_2, α_3 be the distances of the centres of the inscribed and escribed circles respectively from the angle A, and p the perpendicular from A on BC; prove that

$$\alpha\alpha_1\alpha_2\alpha_3 = 4R^2p^2,$$
$$\alpha_2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 16R^2,$$
$$\frac{1}{\alpha^2} + \frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} = \frac{4}{p^2}.$$

6. In any triangle $r^2 + s^2 + 4Rr = bc + ca + ab$.

EXERCISE LX.

1. The sides of a triangle are 7·5, 10, and 12·5. Calculate the value of $\sin 3A + \sin 3B + \sin 3C$.

2. Prove that the radii of the escribed circles are the roots of the equation

$$(x^2 + s^2)(x - r) = 4Rx^2.$$

3. Eliminate ϕ from the equations

$$\rho\cos(\theta - 3\phi) = 2a\cos^3\phi \text{ and } \rho\sin(\theta - 3\phi) = 2a\sin^3\phi.$$

4. P is a point within the triangle ABC; PA, PB, PC equal α, β, γ respectively; r is the radius of the inscribed circle and r_1, r_2, r_3 are the radii of the circles inscribed in PBC, PCA, and PAB. Prove that

$$(r_2+r_3)\,\alpha+(r_3+r_1)\,\beta+(r_1+r_2)\,\gamma \\ =(r-r_1)\,a+(r-r_2)\,b+(r-r_3)\,c.$$

5. Tangents from two points P and Q to a circle whose centre is C make angles α, β; α', β', respectively with the straight line PQ; prove that

$$\frac{\sin\alpha\sin\beta}{\sin\alpha'\sin\beta'}=\frac{CQ^2}{CP^2}.$$

6. If $$a=500,\ b=893,\ c=1167,$$
find $$\cos A+\cos B+\cos C.$$

Application of Theory of Equations.

Identities of a peculiar class may be verified by the theory of equations. To illustrate this we will work out an easy example.

Prove

$$\sin^3(\delta-\beta)\sin^3(\alpha-\gamma)+\sin^3(\beta-\gamma)\sin^3(\alpha-\delta) \\ +\sin^3(\gamma-\delta)\sin^3(\alpha-\beta) \\ =3\sin(\delta-\beta)\sin(\alpha-\gamma)\sin(\beta-\gamma)\sin(\gamma-\delta) \\ \times\sin(\alpha-\delta)\sin(\alpha-\beta).$$

By Newton's theorem for the sums of the powers of the roots of an equation, we have

$$S_m+p_1S_{m-1}+p_2S_{m-2}+\dots+p_{m-1}S_1+mp_m=0,$$
where S_m denotes the sum of the mth powers of the roots.

Consider the expressions $\sin(\delta-\beta)\sin(\alpha-\gamma)$,
$\sin(\beta-\gamma)\sin(\alpha-\delta)$, and $\sin(\gamma-\delta)\sin(\alpha-\beta)$.

It will be an ordinary exercise to prove that their sum is zero. If we make a cubic equation having them for roots, it will therefore be of the form

$$x^3+p_2x+p_3=0.$$

Applying Newton's theorem we have

$$S_1+p_1=0, \quad \therefore S_1=0,$$
$$S_2+b_1S_1+2p_2=0, \; S_2=-2p_2,$$
$$S_3+p_1S_2+p_2S_1+3p_3=0,$$

or

$$S_3+3p_3=0.$$

But $p_3=$ product of the roots with their signs changed; hence the theorem is established.

This particular example may be solved very simply as follows. Let the expressions $\sin(\delta-\beta)\sin(\alpha-\gamma)$, &c. be denoted by x, y, z. Then we have

$$x+y+z=0.$$

Cubing, $$(x+y)^3+z^3+3z(x+y)(x+y+z)=0;$$
$$\therefore x^3+y^3+z^3+3xy(x+y)=0,$$

or

$$x^3+y^3+z^3=3xyz.$$

Expansion of $\sin\theta$ *and* $\cos\theta$ *in powers of* θ.

By De Moivre's Theorem,

$$\cos\theta+\sqrt{-1}\sin\theta=(\cos 1+\sqrt{-1}\sin 1)^\theta$$
$$=A^\theta \text{ where } A=\cos 1+\sqrt{-1}\sin 1,$$

therefore changing the sign of θ

$$\cos\theta - \sqrt{-1}\sin\theta = A^{-\theta},$$

$$\therefore 2\sqrt{-1}\sin\theta = A^{\theta} - A^{-\theta}$$

$$= \left\{1 + \theta\log A + \frac{\theta^2}{\lfloor 2}(\log A)^2 + \frac{\theta^3}{\lfloor 3}(\log A)^3 + \dots\right\}$$

$$- \left\{1 - \theta\log A + \frac{\theta^2}{\lfloor 2}(\log A)^2 - \frac{\theta^3}{\lfloor 3}(\log A)^3 + \dots\right\},$$

by the exponential theorem,

$$= 2\left\{\theta\log A + \frac{\theta^3}{\lfloor 3}(\log A)^3 + \dots\right\};$$

$$\therefore \sqrt{-1}\,\frac{\sin\theta}{\theta} = \log A + \frac{\theta^2}{\lfloor 3}(\log A)^3 + \dots$$

$\therefore$ diminishing θ without limit, we have $\sqrt{-1} = \log A$ or $A = e^{\sqrt{-1}}$

$$\therefore \cos\theta + \sqrt{-1}\sin\theta = A^{\theta}$$

$$= e^{\theta\sqrt{-1}}$$

$$= 1 + \theta\sqrt{-1} + \frac{1}{\lfloor 2}(\theta\sqrt{-1})^2 + \frac{1}{\lfloor 3}(\theta\sqrt{-1})^3 + \dots\dots$$

by the exponential theorem,

$$= 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \dots + \sqrt{-1}\left(\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \dots\right),$$

$\therefore$ equating real and imaginary parts,

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \&c.$$

$$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \&c.$$

This proof has the advantage of not assuming any limit except $\frac{\sin\theta}{\theta}=1$ when $\theta=0$. The series are convergent, because they are deduced directly from the expansion of a^x (by the exponential theorem), which is known to be so.

Imaginary Logarithms of Unity.

Since $$\cos\theta+\sqrt{-1}\sin\theta=e^{\theta\sqrt{-1}},$$
we have (1) $$\log(\cos\theta+\sqrt{-1}\sin\theta)=\theta\sqrt{-1}.$$

Put $\theta=2n\pi$, and this becomes
$$\log 1=2n\pi\sqrt{-1}.$$

We thus obtain an infinite number of logarithms of unity by assigning various values to n. They will be all imaginary, except when $n=0$.

Since any number $a=a\times 1$, we have
$$\log a=\text{ordinary logarithm}+2n\pi\sqrt{-1}.$$

Hence we are led to the theorem, that *every* quantity has an infinite number of logarithms to the base e, obtained by adding $2n\pi\sqrt{-1}$ to the ordinary expression. In the case of any other base b, the only change in the preceding investigation will be that in (1) $\log_b e$ will appear on the right-hand side as a factor. We can thus extend the result to any base.

When we are dealing with real logarithms, we must put n zero, but if a is unreal, the value of n will depend on the special case. The student will be liable to many

fallacies, if when a is unreal, he neglects the imaginary part $2n\pi\sqrt{-1}$ of the logarithm. For example, since $\sin\pi = 0$, we have $e^{\pi\sqrt{-1}} = e^{-\pi\sqrt{-1}}$, or $e^{2\pi\sqrt{-1}} = 1$. Taking logs, the student can *not* infer that $2\pi\sqrt{-1} = 0$.

Reduction of Imaginary Expressions to a simpler form.

This is usually effected by the use of De Moivre's theorem and the exponential values of the sine and cosine. As an example of the method we will solve the following: Reduce the expression $(a + b\sqrt{-1})^{p+q\sqrt{-1}}$ to the form $A + B\sqrt{-1}$.

Assume $\quad a = r\cos\theta, \qquad b = r\sin\theta,$

then $\qquad r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\dfrac{b}{a}$.

Using these values:

$$\begin{aligned}(a + b\sqrt{-1})^{p+q\sqrt{-1}} &= r^{p+q\sqrt{-1}}(\cos\theta + \sqrt{-1}\sin\theta)^{p+q\sqrt{-1}} \\ &= r^{p+q\sqrt{-1}}\,(e^{\theta\sqrt{-1}})^{p+q\sqrt{-1}} \\ &= r^{p+q\sqrt{-1}}\,e^{p\theta\sqrt{-1}-q\theta} \\ &= r^p \,.\, e^{-q\theta}\,e^{(p\theta+q\log r)\sqrt{-1}} \\ &= r^p \,.\, e^{-q\theta}\{\cos(p\theta + q\log r) \\ &\qquad + \sqrt{-1}\sin(p\theta + q\log r)\}.\end{aligned}$$

The required reduction has thus been effected.

Should the given expression be not algebraical, but trigonometrical, *e.g.* $\sin(\alpha + \beta\sqrt{-1})$, it is convenient to assume it equal to $x + y\sqrt{-1}$. Then after substituting the exponential value of the sine and simplifying, we shall obtain two equations to determine x and y, since the real and unreal parts must vanish separately.

Or we may proceed as follows :—

Since $\sin(\alpha+\beta\sqrt{-1}) = x + y\sqrt{-1}$

we have $\sin(\alpha-\beta\sqrt{-1}) = x - y\sqrt{-1}$;

$$\therefore 2x = \sin(\alpha+\beta\sqrt{-1}) + \sin(\alpha-\beta\sqrt{-1}),$$

and $2y\sqrt{-1} = \sin(\alpha+\beta\sqrt{-1}) - \sin(\alpha-\beta\sqrt{-1})$.

On simplifying, $\sqrt{-1}$ will disappear from these expressions for x and y.

Values of certain Symmetrical Functions.

To find the value of any symmetrical function of the quantities whose general term is $\cos\left(\alpha+\frac{2r\pi}{n}\right)$ *or* $\sin\left(\alpha+\frac{2r\pi}{n}\right)$, *where* r *takes in succession all integral values from* 0 *to* $n-1$.

This may be done (as explained in the Theory of Equations) if we can find an equation whose roots are the above quantities.

There are four cases, according as cosines or sines are involved, and also as n is even or odd.

We know that, when n is even,

$$(-1)^{\frac{n}{2}} \cos n\theta = 1 - \frac{n^2}{\underline{|2}}\cos^2\theta + \frac{n^2(n^2-2^2)}{\underline{|4}}\cos^4\theta - \ldots\ldots$$
$$+ (-1)^{\frac{n}{2}}\, 2^{n-1}\cos^n\theta.$$

Substitute $\cos\left(\alpha+\frac{2r\pi}{n}\right)$ for $\cos\theta$; then since $\cos n\theta = \cos(n\alpha+2r\pi) = \cos n\alpha$ for every integral value of r, we

see that $\cos\left(\alpha+\frac{2r\pi}{n}\right)$ is a root of the equation

$$(-1)^{\frac{n}{2}}\cos n\alpha = 1 - \frac{n^2}{\lfloor 2}\cos^2\theta + \frac{n^2(n^2-2^2)}{\lfloor 4}\cos^4\theta - \dots\dots$$
$$+(-1)^{\frac{n}{2}}\,2^{n-1}\cos^n\theta \dots\dots\dots\dots (1),$$

the n *different* roots being obtained by making r equal (in succession) to $0, 1, 2, \dots\dots n-1$.

In like manner, we may show that in the other cases the required equations are

$$\cos n\alpha = 1 - \frac{n^2}{\lfloor 2}\sin^2\theta + \frac{n^2(n^2-2^2)}{\lfloor 4}\sin^4\theta - \dots\dots$$
$$+(-1)^{\frac{n}{2}}\,2^{n-1}\sin^n\theta \dots\dots\dots\dots (2),$$

$$(-1)^{\frac{n-1}{2}}\cos n\alpha = n\cos\theta - \frac{n(n^2-1^2)}{\lfloor 3}\cos^3\theta + \dots\dots\dots\dots$$
$$+(-1)^{\frac{n-1}{2}}\,2^{n-1}\cos^n\theta \dots\dots\dots (3),$$

$$\sin n\alpha = n\sin\theta - \frac{n(n^2-1^2)}{\lfloor 3}\sin^3\theta + \dots\dots\dots\dots$$
$$+(-1)^{\frac{n-1}{2}}\,2^{n-1}\sin^n\theta \dots\dots\dots\dots (4),$$

n being *even* in (1) and (2), and *odd* in (3) and (4).

For example, to find the sum of the products taken four together, of $\cos\alpha$, $\cos\left(\alpha+\frac{\pi}{3}\right), \dots\dots \cos\left(\alpha+\frac{5\pi}{3}\right)$.

These quantities are the roots of (1) when $n=6$; i.e. they are the roots of

$$32\cos^6\theta + \dots\dots + 18\cos^2\theta - 1 - \cos n\alpha = 0.$$

The sum required, by the Theory of Equations, is $\frac{18}{32}$ or $\frac{9}{16}$.

As another example, find the product of

$$\sin\alpha,\ \sin\left(\alpha+\frac{2\pi}{n}\right),\ \ldots\ldots\ \sin\left(\alpha+\frac{2(n-1)\pi}{n}\right).$$

When n is even, we have from (2)

$$\text{Product} = \frac{\text{constant term}}{\text{coefficient of } \sin^n\theta}$$

$$= \frac{1-\cos n\alpha}{(-1)^{\frac{n}{2}}\, 2^{n-1}}.$$

When n is odd, we have from (4)

$$\text{Product} = -\frac{\text{constant term}}{\text{coefficient of } \sin^n\theta}$$

$$= \frac{\sin n\alpha}{(-1)^{\frac{n-1}{2}}\, 2^{n-1}}.$$

In this class of examples, the terms near the beginning and end of the equation are usually the only ones required. These may readily be obtained without knowing the general term.

For instance :—We know that when n is odd,

$$\sin n\theta = A_1 \sin\theta + A_3 \sin^3\theta + \ldots\ldots + A_{n-2}\sin^{n-2}\theta + A_n \sin^n\theta.$$

To find the terms near the beginning, expand the sines in power of θ. Thus we have

$$n\theta - \frac{n^3\theta^3}{\lfloor 3} + \frac{n^5\theta^5}{\lfloor 5} - \ldots = A_1\left(\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \ldots\right)$$
$$+ A_3\left(\theta - \frac{\theta^3}{\lfloor 3} + \ldots\right)^3 + \ldots$$

$\therefore$ equating coefficients $A_1 = n$,

$$A_3 - \frac{A_1}{6} = -\frac{n^3}{6} \text{ or } A_3 = \frac{n - n^3}{6},$$

and so on.

To find the terms near the end, substitute $\dfrac{x - \frac{1}{x}}{2\sqrt{-1}}$ for $\sin\theta$. We have

$$\frac{x^n - \frac{1}{x^n}}{2\sqrt{-1}} = A_1 \frac{x - \frac{1}{x}}{2\sqrt{-1}} + \ldots\ldots$$

$$+ A_{n-2}\left(\frac{x - \frac{1}{x}}{2\sqrt{-1}}\right)^{n-2} + A_n\left(\frac{x - \frac{1}{x}}{2\sqrt{-1}}\right)^n.$$

$\therefore$ equating coefficients

$$\frac{A_n}{(2\sqrt{-1})^n} = \frac{1}{2\sqrt{-1}} \text{ or } A_n = (-1)^{\frac{n-1}{2}} 2^{n-1},$$

$$\frac{A_{n-2}}{(2\sqrt{-1})^{n-2}} - \frac{nA_n}{(2\sqrt{-1})^n} = 0 \text{ or } A_{n-2} = (-1)^{\frac{n-3}{2}} n 2^{n-3},$$

and so on.

Expansion of sin θ *and* cos θ *in factors.*

The rigorous proof given in Todhunter's *Trigonometry* of the mode of breaking $\sin\theta$ and $\cos\theta$ into factors is very difficult to remember. The other proof may be made more satisfactory as follows:

Lemma. The equation $\sin x = 0$ cannot be satisfied by imaginary values of x. For, let $\alpha + \beta\sqrt{-1}$ be such a root. Then

$$\sin(\alpha + \beta\sqrt{-1}) = 0,$$

or $$e^{(\alpha+\beta\sqrt{-1})\sqrt{-1}} - e^{-(\alpha+\beta\sqrt{-1})\sqrt{-1}} = 0,$$
which gives $$e^{2\alpha\sqrt{-1}-2\beta} = 1.$$

[In taking logarithms, the student may be in danger of concluding that $\alpha\sqrt{-1} = \beta$ which is absurd, since β is real by hypothesis. When the exponential is imaginary, we cannot neglect the imaginary logarithms of unity.]

Hence we have
$$2\alpha\sqrt{-1} - 2\beta + 2n\pi\sqrt{-1} = 0.$$
Since the real parts and the imaginary must vanish separately, we have $\beta = 0$. Thus the Lemma is proved.

We know that $\sin\theta$ may be expanded in a *convergent* series of powers of θ, viz. $\theta - \frac{\theta^3}{\lfloor 3} + \&c.$, and that this series vanishes when $\theta = 0$, $\theta = \pm\pi$, &c. Hence the series must be divisible by θ, $\theta + \pi$, $\theta - \pi$, &c., and we may therefore assume that

$\sin\theta =$ constant $\times$ product of factors corresponding to the roots
$$= A\theta(\theta+\pi)(\theta-\pi)(\theta+2\pi)(\theta-2\pi)\ldots\ldots$$
where A contains the product of the imaginary roots.

But we know by the previous Lemma that there are not any imaginary roots, and therefore that A does not contain θ.

Therefore $\sin\theta = A\theta(\theta^2 - \pi^2)(\theta^2 - 2^2\pi^2)\ldots\ldots$
or, altering the constant
$$\sin\theta = a\theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\ldots\ldots$$
Diminishing θ indefinitely, we find that $a = 1$.

A similar method may be used to break up $\cos\theta$ into its factors.

Summation of certain series.

The sums of many series may be deduced from the expansions of $\sin\theta$ and $\cos\theta$.

We have $\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots\ldots$

and also $\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \ldots\ldots$

These series must be identical, and by equating the coefficients of like powers of θ, we obtain sums of series. The most convenient mode of doing this will be to take logarithms before comparing coefficients. We shall have then

$$\log\frac{\sin\theta}{\theta} = \log\left(1-\frac{\theta^2}{\pi^2}\right) + \log\left(1-\frac{\theta^2}{2^2\pi^2}\right) + \&c.$$

$$= \log\left(1 - \frac{\theta^2}{\underline{|3}} + \frac{\theta^4}{\underline{|5}} - \&c.\right).$$

Expanding the logarithms by the series for $\log(1-x)$, we can at once equate coefficients of low powers of θ.

By using the Differential Calculus, we may obtain other interesting results.

$$\log\sin\theta = \log\theta + \log\left(1-\frac{\theta^2}{\pi^2}\right) + \ldots$$

Differentiating $\cot\theta = \frac{1}{\theta} - \frac{2\theta}{\pi^2-\theta^2} - \frac{2\theta}{2^2\pi^2-\theta^2} - \&c.$

a similar series may be found for $\tan\theta$.

By assigning special values to θ, many numerical relations may be established. These values need not be necessarily real. For instance, in the above identity, we

may write for θ, $\pi\sqrt{-1}$. Substituting for $\cot\theta$ its exponential value, the imaginary expression $\sqrt{-1}$ will disappear, and we shall be left with the numerical identity

$$\pi\frac{\epsilon^{\pi}+\epsilon^{-\pi}}{\epsilon^{\pi}-\epsilon^{-\pi}}=1+\frac{2}{1^2+1}+\frac{2}{2^2+1}+\&c.$$

EXERCISE LXI.

1. Defining $\cos\theta$ as the real part of $e^{\theta\sqrt{-1}}$ and $\sqrt{-1}\sin\theta$ as the unreal part, obtain the fundamental equations

$$\sin^2 A+\cos^2 A=1,$$
$$\sin(A+B)=\sin A\cos B+\cos A\sin B.$$

2. The sides of a triangle are 13, 17, 22; find the perimeter of the triangle joining the feet of perpendiculars.

3. Prove that $32\operatorname{cosec}^5 2\alpha-\tan^5\alpha-\cot^5\alpha$

$$=\tfrac{5}{27}(8\operatorname{cosec}^3 2\alpha-\tan^3\alpha-\cot^3\alpha)^3$$
$$-\tfrac{5}{3}(8\operatorname{cosec}^3 2\alpha-\tan^3\alpha-\cot^3\alpha)$$
$$=\tfrac{5}{3}\{8\operatorname{cosec}^3 2\alpha-\tan^3\alpha-\cot^3\alpha\}$$
$$\times\{4\operatorname{cosec}^2 2\alpha+\tan^2\alpha+\cot^2\alpha\}.$$

4. If $\tan\phi'=\dfrac{(1+m)\tan\phi}{1-m\tan^2\phi}$, and $\tan\phi''=\dfrac{(1-m)\tan\phi}{1+m\tan^2\phi}$, express m in terms of ϕ' and ϕ''.

5. Evaluate $\dfrac{(6x-8\sin x+\sin 2x)^4}{(3-4\cos x+\cos 2x)^5}$ when $x=0$.

6. If x, y, z, be the distances of the centre of the inscribed circle from A, B, C, prove

$$x\cos\frac{A}{2}+y\cos\frac{B}{2}+z\cos\frac{C}{2}=4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

EXERCISE LXII.

1. Prove that

$$\sin 5^\circ + \sin 15^\circ \cos 20^\circ + \sin 25^\circ \cos 60^\circ$$
$$+ \sin 35^\circ \cos 120^\circ + \sin 45^\circ \cos 200^\circ + \sin 55^\circ \cos 300^\circ = 0.$$

2. The hypothenuse of a right-angled triangle is 12·5, and the sum of the other sides is 17·5. Find the value of r.

3. O is the point within ABC at which the sides subtend equal angles. Prove that

$$a \sin(B-C).OA + b \sin(C-A).OB + c \sin(A-B).OC = 0.$$

4. Sum to n terms

$$\sin x + 2 \sin 2x + 3 \sin 3x + \ldots\ldots$$

5. If α be an imaginary cube root of unity, prove that one value of $\alpha^{\frac{1}{3}} + \alpha^{\frac{2}{3}}$ is $2 \sin 50^\circ$.

6. If $\dfrac{u-a}{\cos\theta} = \dfrac{v-b}{\sin\theta} = c$, and $\dfrac{u-a'}{\cos\theta'} = \dfrac{v-b'}{\sin\theta'} = c'$, shew that $(c+c')^2$ $(a-a')^2 + (b-b')^2$ and $(c-c')^2$ are in descending or ascending order of magnitude, according as c and c' have the same or opposite signs.

EXERCISE LXIII.

1. Prove that $\dfrac{\sin\alpha}{1-\cos\alpha} = \cot\dfrac{\alpha}{2}$ by exponential values.

2. If $$\tan(\psi - \phi) = \cos\alpha \tan\phi,$$ shew that $(1 - \sin^2\alpha \sin^2\phi)\left(1 - \tan^4\dfrac{\alpha}{2}\sin^2\psi\right)$ is a perfect square.

3. Given a, b, C. If small errors α, β, are made in measuring the sides, find the error in the computed value of the perpendicular from C on AB.

4. The values of $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{p}}$ are got by putting $r=0, 1, 2, \ldots p-1$ in

$$\cos\frac{2r\pi+\theta}{p}+\sqrt{-1}\sin\frac{2r\pi+\theta}{p}.$$

Shew that the product of any two values equidistant from the beginning and end of this succession is constant for all such pairs.

5. If p, q, r are the lengths of the bisectors of the angles of a triangle, and p', q', r' their lengths when produced to meet the circumscribing circle, prove that

$$(1)\quad \frac{\cos\frac{A}{2}}{p}+\frac{\cos\frac{B}{2}}{q}+\frac{\cos\frac{C}{2}}{r}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

$$(2)\quad p'\cos\frac{A}{2}+q'\cos\frac{B}{2}+r'\cos\frac{C}{2}=a+b+c.$$

EXERCISE LXIV.

1. Given $\tan\theta\tan\phi=\left\{\frac{(m-1)(m+3)}{(m+1)(3-m)}\right\}^2$,

$$\frac{\tan\theta}{\tan\phi}=\frac{(m+1)(m+3)}{(m-1)(3-m)},$$

find the value of $\sqrt{\sin\theta\sin\phi}+\sqrt{\cos\theta\cos\phi}$.

2. If $$3\tan\phi=2\tan\frac{\omega+\phi}{2},$$

prove that $$\cos\omega=\frac{(A+B\sin^2\phi)\cos\phi}{C+D\sin^2\phi},$$

where A, B, C, D, are constants to be determined.

3. Shew that

$$16 \sin 20^\circ \sin 40^\circ \sin 50^\circ \sin 70^\circ = 1 + \alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}},$$

where α and β are the imaginary cube roots of unity.

4. Put into a known rational form the expression $\sqrt{-1} \cos^{-1} \frac{x}{a}$ when x is greater than a.

5. Eliminate θ and ϕ from

$$\cos\theta + \cos\phi = \frac{x}{a},$$

$$\sin\theta + \sin\phi = \frac{y}{b},$$

$$a^2 (\cos\theta - \cos\phi)^2 + b^2 (\sin\theta - \sin\phi)^2 = c^2.$$

6. Find the sum of

$$\sin\frac{2\pi}{n+2} + \sin\frac{4\pi}{n+2} + \sin\frac{6\pi}{n+2} + \ldots\ldots + \sin\frac{2n\pi}{n+2}.$$

EXERCISE LXV.

1. If Δ be the area of the triangle ABC, and Δ' that of the triangle formed by joining the points in which the bisectors of the angles meet the opposite sides, then

$$\frac{\Delta'}{\Delta} = \frac{2 \sin A \sin B \sin C}{(\sin B + \sin C)(\sin C + \sin A)(\sin A + \sin B)}.$$

2. Evaluate $\dfrac{m \sin\theta - \sin m\theta}{\theta(\cos\theta - \cos m\theta)}$ when $\theta = 0$.

3. Find the real and imaginary parts of

$$\cos(a + b\sqrt{-1}).$$

4. Solve for θ in the equation

$$\frac{\sin(\alpha+\theta)}{\sqrt{a^2 + b^2 - 2ab\cos(\alpha+\theta)}} = \frac{\sin(\alpha-\theta)}{\sqrt{a^2 + b^2 - 2ab\cos(\alpha-\theta)}}.$$

5. O is the centre of the circumscribing circle, and I_1 that of the escribed circle touching BC; if O, I_1, B, C, lie on a circle, shew that

$$\text{area } BOCI_1 = \frac{a^2}{4\sqrt{3}}(1+n), \text{ where } n = \frac{a+b+c}{a}.$$

EXERCISE LXVI.

1. Prove that

$$\cot\theta = \tan\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta.$$

2. If $a = 40$, $b = 60$, $c = 80$, find the lengths of the *medians to five places of decimals.

3. Find all the values of θ determined by

$$\sin\theta + \sin 3\theta + \sin 5\theta + \ldots\ldots + \sin(2n-1)\theta$$
$$= \cos\theta + \cos 3\theta + \cos 5\theta + \ldots\ldots + \cos(2n-1)\theta.$$

4. Assuming De Moivre's Theorem, prove that $\cos\theta + \sqrt{-1}\sin\theta = e^{\theta\sqrt{-1}}$ and hence shew that

$$\sin^{-1}(a\sqrt{-1}) = \sqrt{-1}\log(a + \sqrt{1+a^2}).$$

5. Shew that two series in Geometrical Progression can be formed by taking one root of each of the following n equations:

$$x^2 - 2x\cos\theta + 1 = 0,$$
$$x^2 - 2x\cos 2\theta + 1 = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x^2 - 2x\cos n\theta + 1 = 0,$$

and that the product of the two series is

$$\sin^2\frac{n\theta}{2}\operatorname{cosec}^2\frac{\theta}{2}.$$

* See page 8.

EXERCISE LXVII.

1. If $\tan\theta = \sqrt{-1}$, prove that $\tan n\theta = \sqrt{-1}$.

2. If $A + B + C = 180^\circ$, and

$$l = \frac{\sin A + \sin B + \sin C}{\sin 2A + \sin 2B + \sin 2C},$$

$$m = \frac{-\sin A + \sin B + \sin C}{-\sin 2A + \sin 2B + \sin 2C},$$

$$n = \frac{\sin A - \sin B + \sin C}{\sin 2A - \sin 2B + \sin 2C},$$

$$r = \frac{\sin A + \sin B - \sin C}{\sin 2A + \sin 2B - \sin 2C},$$

express the product $lmnr$ in terms of $\cos A$, $\cos B$, $\cos C$.

3. From the known sum of

$$\sin\alpha + \sin(\alpha+\theta) + \sin(\alpha + 2\theta) + \ldots\ldots \text{ to } n \text{ terms},$$

deduce the sum of n terms of the Arithmetical Progression

$$a + (a+b) + (a+2b) + \ldots\ldots, \text{ to } n \text{ terms}.$$

4. Prove that

$$\sin^{-1}\frac{m}{\sqrt{m^2+n^2}} + \sin^{-1}\frac{n-m}{\sqrt{2(m^2+n^2)}} = (p \pm \tfrac{1}{4})\pi,$$

where p is any integer.

5. Prove that

$$\tan^{-1}\left(\frac{\tan 2\theta + \tanh 2\phi}{\tan 2\theta - \tanh 2\phi}\right) + \tan^{-1}\left(\frac{\tan\theta - \tanh\phi}{\tan\theta + \tanh\phi}\right)$$
$$= \tan^{-1}(\cot\theta \coth\phi),$$

where tanh and coth are defined as in Exercise LVIII.

EXERCISE LXVIII.

1. Solve by Trigonometry

$$x^3 + 3x^2 = \frac{5+\sqrt{5}}{2}.$$

2. If $$\tan\frac{\theta}{2}\tan\frac{\phi}{2}=\frac{b-c}{b+c},$$
prove that
$$\frac{b^2+c^2-2bc\cos\theta}{\sin^2\theta}=\frac{b^2+c^2-2bc\cos\phi}{\sin^2\phi}$$
and $(b^2+c^2-2bc\cos\theta)(b^2+c^2-2bc\cos\phi)=(b^2-c^2)^2$.

3. Prove that
$$\tan n\theta=\frac{\sin\theta+\sin 3\theta+\sin 5\theta+\ldots\ldots+\sin(2n-1)\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\ldots\ldots+\cos(2n-1)\theta}.$$

4. Sum to infinity
$$\cos\theta+\tfrac{1}{2}\cos 2\theta+\tfrac{1}{3}\cos 3\theta+\ldots\ldots$$

5. Find the real and imaginary parts of
$$\sec(\alpha+\beta\sqrt{-1}).$$

6. If S_1, S_1', S_2, S_2', S_3, S_3', be the segments of the sides of a triangle made by the bisectors of the angles, prove that
$$S_1 S_2 S_3=S_1' S_2' S_3'=\frac{a^2b^2c^2}{(a+b)(b+c)(c+a)}.$$

EXERCISE LXIX.

1. If $2\cos\theta=(\sqrt{-1})^n e^{cx\sqrt{-1}}+(-\sqrt{-1})^n e^{-cx\sqrt{-1}}$, find θ.

2. Given that
$$\cos\omega=\frac{\cos\phi\,(4-\sin^2\phi)}{4+5\sin^2\phi},$$
find the value of $$\frac{\tan\dfrac{\omega+\phi}{2}}{\tan\phi}.$$

3. Sum to n terms

$$1 + 3\cos\theta + 5\cos 2\theta + 7\cos 3\theta + \ldots\ldots$$

4. Find the sum of an infinite number of terms of the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$

when all the terms $\left(\frac{1}{2m}\right)^2$, $\left(\frac{1}{3m}\right)^2$, $\left(\frac{1}{4m}\right)^2$, ... are removed.

5. ABC is a triangle, I the centre of the inscribed circle, A', B', C' the centres of the circles circumscribed about the triangles BIC, CIA, AIB. Prove

(1) that AA' BB' CC' all pass through I.

(2) the triangles ABC, $A'B'C'$ have the same circumscribing circle.

(3) if Δ, Δ' be the areas of the triangles ABC, $A'B'C'$, then $\frac{\Delta}{\Delta'} = \frac{2r}{R}$, where r, R are the radii of the inscribed and circumscribed circles of the triangle ABC.

6. If $$\frac{1 - \sin\tau}{1 + \sin\tau} = \left(\frac{1 - \sin\sigma}{1 + \sin\sigma}\right)^5,$$

express $$\frac{1}{1 - \cos^2\frac{\pi}{10}\cos^2\sigma} + \frac{1}{1 - \cos^2\frac{3\pi}{10}\cos^2\sigma}$$

in terms of $\frac{\sin\tau}{\sin\sigma}$.

EXERCISE LXX.

1. Shew that, if $\alpha+\beta+\gamma=\frac{\pi}{4}$,

$$\frac{\cos 2\alpha \sin(\beta-\gamma)+\cos 2\beta \sin(\gamma-\alpha)+\cos 2\gamma \sin(\alpha-\beta)}{\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}}$$
$$=16\cos\left(\frac{\pi}{8}+\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{8}+\frac{\beta}{2}\right)\cos\left(\frac{\pi}{8}+\frac{\gamma}{2}\right).$$

2. Find the value of

$$\frac{\tan 2\theta-2\tan\theta}{\theta^3} \text{ when } \theta=0.$$

3. Prove that

$$(x+4)(x-1)=3\log x+\tfrac{7}{2}(\log x)^2+\tfrac{13}{6}(\log x)^3+\ldots$$

4. Prove that

$$\cos\frac{\pi}{25}+\cos\frac{2\pi}{25}+\cos\frac{3\pi}{25}+\ldots\ldots+\cos\frac{20\pi}{25}$$
$$=4\cos\frac{\pi}{10}\cos\frac{\pi}{25}\cos\frac{\pi}{50}.$$

5. Prove that

$$\frac{\log(-1)}{\sqrt{-1}}=(2n+1)\pi$$

where n is any integer.

6. Prove that

$$\begin{vmatrix} 1, & 1, & 1 \\ \cos^4\alpha, & \cos^4\beta, & \cos^4\gamma \\ \sin^4\alpha, & \sin^4\beta, & \sin^4\gamma \end{vmatrix}$$
$$=\tfrac{1}{4}\{\cos 2\gamma-\cos 2\beta\}\{\cos 2\beta-\cos 2\alpha\}\{\cos 2\alpha-\cos 2\gamma\}.$$

EXERCISE LXXI.

1. If $P = 4\sin^2\theta\sin^2\phi\cos^2(\theta+\phi) - (\sin^2\theta+\sin^2\phi)^2 + 2\sin^2(\theta+\phi)\{\sin^2\theta+\sin^2\phi\}$,

express $\sqrt[4]{P}$ in its simplest form.

2. Prove that

$$\frac{\sin x}{1-2a\cos x+a^2} = \sin x + a\sin 2x + a^2\sin 3x + \ldots\ldots$$

3. Shew that

$$n\sin\theta + (n-1)\sin 2\theta + (n-2)\sin 3\theta + \ldots\ldots + \sin n\theta$$
$$= \frac{n+1}{2}\cot\frac{\theta}{2} - \frac{\sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}.$$

4. Sum to infinity

(1) $$\cos\alpha + \frac{x}{\underline{|1}}\cos(\alpha+\beta) + \frac{x^2}{\underline{|2}}\cos(\alpha+2\beta) + \frac{x^3}{\underline{|3}}\cos(\alpha+3\beta) + \ldots\ldots$$

(2) $$\sin\alpha + \frac{x}{\underline{|1}}\sin(\alpha+\beta) + \frac{x^2}{\underline{|2}}\sin(\alpha+2\beta) + \frac{x^3}{\underline{|3}}\sin(\alpha+3\beta) + \ldots\ldots$$

5. From any triangle a portion is cut off by a tangent to the inscribed circle, and, from the triangle so formed, another is cut off similarly *ad inf.*, all the tangents being parallel to BC. Prove that the sum of the areas

$$= \frac{\Delta(s-a)^2}{a(b+c)}.$$

EXERCISE LXXII.

1. Prove that

$$\left\{1+\tan\frac{\pi+\theta}{18}\tan\frac{11\pi+2\theta}{36}\right\}\left\{\tan\frac{\pi+\theta}{18}+\tan\frac{5\pi-\theta}{18}\right\}^2$$
$$=\frac{3\left\{\tan\frac{11\pi+2\theta}{36}-\tan\frac{\pi+\theta}{18}\right\}}{\left\{1+\cos\frac{\pi+\theta}{9}\right\}\left\{1+\cos\frac{5\pi-\theta}{9}\right\}}.$$

2. Find the limiting value of

$$\frac{\tan nx-\tan x}{n\sin x-\sin nx}$$

when $x=0$.

3. In a triangle

$$a^2=b^2+c^2-2bc\cos A;$$

find the corresponding relation in a polygon.

4. Eliminate θ and ϕ between the equations

$$x\cos\theta+y\sin\theta=b,$$
$$ax\cos\phi+by\sin\phi=0,$$
$$a\cos\phi\cos\theta+b\sin\phi\sin\theta=b.$$

5. In a given triangle ABC let the triangle joining the feet of perpendiculars be called the first triangle; let another such triangle be formed in this last and be called the second triangle, and so on. If s_n, A_n, B_n, C_n be the semi-perimeter and angles respectively of the nth triangle, shew that the radius of the circle circumscribing ABC is

$$\frac{2^n s_n}{\sin A_n+\sin B_n+\sin C_n}.$$

6. Sum to n terms the Geometrical Progression

$$\{\cos\theta + \sqrt{-1}\sin\theta\} + \{\cos\theta + \sqrt{-1}\sin\theta\}^2 + \ldots\ldots$$

and deduce the sums of

$$\cos\theta + \cos 2\theta + \ldots\ldots + \cos n\theta$$

and

$$\sin\theta + \sin 2\theta + \ldots\ldots + \sin n\theta.$$

EXERCISE LXXIII.

1. Shew that in any triangle

$$\begin{aligned} &a\cos A\cos 2A\cos 4A\ldots\ldots\cos 2^{n-1}A\cos 2^n C \\ &\quad + c\cos C\cos 2C\cos 4C\ldots\ldots\cos 2^{n-1}C\cos 2^n A \\ &\quad\quad + b\cos B\cos 2B\cos 4B\ldots\ldots\cos 2^{n-1}B = 0. \end{aligned}$$

2. Find the circular measure of x to six places of decimals, where

$$\sin\left(x + \frac{\pi}{6}\right) = 10\sin x.$$

3. Evaluate, when $x = \frac{\pi}{2}$,

(1) $x\tan x - \frac{\pi}{2}\sec x$,

(2) $\frac{\pi}{2}\tan x - x\sec x$.

4. Sum to infinity the series

$$\cos\theta\cos 2\theta + \frac{1}{\lfloor 1}\cos 2\theta\cos 3\theta + \frac{1}{\lfloor 2}\cos 3\theta\cos 4\theta + \ldots\ldots$$

the general term being $\frac{1}{\lfloor n}\cos(n+1)\theta\cos(n+2)\theta$.

5. Separate

$$(a + b\sqrt{-1})\sin(\alpha + \beta\sqrt{-1})$$

into its real and unreal parts.

EXERCISE LXXIV.

1. Prove that in any triangle

$$\frac{\sin 5A \sin (B-C) + \sin 5B \sin (C-A) + \sin 5C \sin (A-B)}{\sin (B-C) \sin (C-A) \sin (A-B)} + 16 \sin A \sin B \sin C = 0.$$

2. If $\dfrac{\sin (x-y)}{\sin (x-z)} = \dfrac{m}{n}$ and $\dfrac{\cos (x-y)}{\cos (x-z)} = \dfrac{p}{q}$,

then will $$\cos (y-z) = \frac{mp+nq}{mq+np}.$$

3. Shew that

$$\sqrt{-1} + (16 + 16\sqrt{-1})^{\frac{1}{3}} = 1 + (1+\sqrt{-1})\sqrt{3}.$$

4. Find the product

$$(\sin\theta \cos \tfrac{1}{2}\theta)^{\frac{1}{2}} (\sin \tfrac{1}{2}\theta \cos \tfrac{1}{4}\theta)^{\frac{1}{4}} (\sin \tfrac{1}{4}\theta \cos \tfrac{1}{8}\theta)^{\frac{1}{8}} \ldots\ldots$$

to n factors and to infinity.

5. If $x = \cos\theta + \sqrt{-1} \sin\theta$, shew that

$$\sin^{-1} x = \tfrac{1}{2} \cos^{-1} (2 \sin\theta - 1) + \tfrac{1}{2} \cos^{-1} (1 + 2 \sin\theta).$$

6. Triangles are inscribed in a fixed circle. Shew that if the inscribed circle of the pedal triangle is a maximum, the inscribed triangle must be equilateral.

EXERCISE LXXV.

1. Prove that

$$\tan^6 \alpha + \cot^6 \alpha - \tan^2 \alpha - \cot^2 \alpha$$
$$= 64 \operatorname{cosec}^4 2\alpha \cot^2 2\alpha - 32 \operatorname{cosec}^2 2\alpha \cot^2 2\alpha$$
$$= 16 \operatorname{cosec}^2 2\alpha \cot^2 2\alpha (\tan^2 \alpha + \cot^2 \alpha).$$

2. Sum to infinity

$$\cos\theta + \frac{\operatorname{cosec}\theta}{\lfloor 1} \cos 2\theta + \frac{\operatorname{cosec}^2\theta}{\lfloor 2} \cos 3\theta + \ldots\ldots$$

3. Of what order is the error incurred when it is assumed that

$$\theta = \frac{3\sin\theta}{2+\cos\theta},$$

θ being a small quantity?

4. Prove that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{2^7}.$$

5. Prove that

$$1+\frac{\pi^4}{\lfloor 4}+\frac{\pi^8}{\lfloor 8}+\ldots=\frac{1}{2}\left\{\left(1+\frac{4}{1^2}\right)\left(1+\frac{4}{3^2}\right)\left(1+\frac{4}{5^2}\right)\ldots ad\ inf.-1\right\}.$$

6. Shew that

$$\cos^{-1}(1+2\sin\theta)=2\sqrt{-1}\log\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}+\sqrt{\sin\theta}\right).$$

EXERCISE LXXVI.

1. Prove that

$$2\log c=\log a+\log b-\frac{a^2+b^2}{ab}\cos C-\frac{1}{2}\frac{a^4+b^4}{a^2b^2}\cos 2C-\ldots.$$

2. Shew that, if $\alpha+\beta+\gamma=\pi$,

$$\begin{vmatrix} 1, & 1, & 1 \\ \cos\alpha, & \cos\beta, & \cos\gamma \\ \cot\alpha, & \cot\beta, & \cot\gamma \end{vmatrix} = -\frac{\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}.$$

3. Determine x from the equation

$$\sin^{-1}\frac{a}{x}+\sin^{-1}\frac{b}{x}+\sin^{-1}\frac{c}{x}=\pi.$$

4. Shew that

$$1+\frac{1}{3}\left(\frac{3}{4}\right)+\frac{1}{5}\left(\frac{3}{4}\right)^2+\frac{1}{7}\left(\frac{3}{4}\right)^3+\ldots\ldots$$

$$=\frac{2}{\sqrt{3}}\log 2+1-\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^2-\frac{1}{5}\left(\frac{\sqrt{3}}{2}\right)^3+\ldots\ldots$$

$$=\frac{1}{\sqrt{3}}\log 7+\frac{4}{7}-\frac{1}{2}\cdot\frac{4}{7}\left(\frac{4\sqrt{3}}{7}\right)+\frac{1}{3}\cdot\frac{4}{7}\left(\frac{4\sqrt{3}}{7}\right)^2$$

$$-\frac{1}{5}\cdot\frac{4}{7}\left(\frac{4\sqrt{3}}{7}\right)^3+\ldots\ldots$$

$$=\frac{2}{\sqrt{3}}-\frac{1}{2}\left\{\frac{2}{\sqrt{3}}+\frac{\sqrt{3}}{2}\right\}+\frac{1}{3}\left\{\frac{2}{\sqrt{3}}+\left(\frac{\sqrt{3}}{2}\right)^2\right\}$$

$$-\frac{1}{5}\left\{\frac{2}{\sqrt{3}}+\left(\frac{\sqrt{3}}{2}\right)^3\right\}+\ldots\ldots.$$

5. Prove that

$$\sin^4(\delta-\beta)\sin^4(\alpha-\gamma)+\sin^4(\beta-\gamma)\sin^4(\alpha-\delta)$$
$$+\sin^4(\gamma-\delta)\sin^4(\alpha-\beta)$$

$$=2\left\{\begin{array}{l}\sin^2(\delta-\beta)\sin^2(\alpha-\gamma)+\sin^2(\beta-\gamma)\sin^2(\alpha-\delta)\\ \qquad\qquad+\sin^2(\gamma-\delta)\sin^2(\alpha-\beta)\end{array}\right\}^2.$$

6. Eliminate x between

$$a\tan(x+\alpha)=b\tan(x+\beta)=c\tan(x+\gamma),$$

expressing the result as a determinant.

EXERCISE LXXVII.

1. Express a and b in terms of c, C, and Δ.

2. Find the limit of

$$\left\{\frac{\cot\theta}{\sqrt{2-2\sin\theta}}\right\}^{\tan^2\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}$$

when $\theta=\frac{\pi}{2}$.

3. Shew that $\cos^n\theta\cos n\theta$

$$=1-\frac{n(n+1)}{\lfloor 2}\tan^2\theta+\frac{n(n+1)(n+2)(n+3)}{\lfloor 4}\tan^4\theta+\ldots.$$

4. Prove that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots\ldots=\frac{6}{\pi^2},$$

where 2, 3, 5, 7, ... are all the prime numbers.

5. Shew that, if A, B, and C are the angles of a triangle,

$$\begin{vmatrix} \cos A, & \cos B, & \cos(C+2A) \\ \cos A, & \cos(B+2C), & \cos C \\ \cos(A+2B), & \cos B, & \cos C \end{vmatrix}=0.$$

6. ABC is a triangle inscribed in a circle, R is any point in the arc AB; a hexagon $ARBPCQ$ is completed, having its opposite sides parallel; also two triangles are formed by producing AR, BP, CQ, and AQ, CP, BR respectively: prove that these triangles are similar to ABC and have their homologous sides parallel; and that the sum of the homologous sides of the two triangles is to the homologous side of ABC as

$$(\sin^2 A+\sin^2 B+\sin^2 C)\sin\theta : \sin A\sin B\sin C,$$

where θ is the angle between the homologous sides of the triangles and that of ABC.

EXERCISE LXXVIII.

1. Prove that

$$\frac{\tan\alpha}{\tan(\alpha-\beta)\tan(\alpha-\gamma)}+\frac{\tan\beta}{\tan(\beta-\alpha)\tan(\beta-\gamma)}$$
$$+\frac{\tan\gamma}{\tan(\gamma-\alpha)\tan(\gamma-\beta)}=\tan\alpha\tan\beta\tan\gamma.$$

2. Solve the equation

$$17 \sin\left(\frac{\pi}{4} - \theta\right) = \frac{3}{\sqrt{2}} \operatorname{cosec} \theta.$$

3. Shew that

$$\tan^{-1} \frac{a \sin 2\theta}{1 - a \cos 2\theta} = a \sin 2\theta + \tfrac{1}{2} a^2 \sin 4\theta + \tfrac{1}{3} a^3 \sin 6\theta + \ldots.$$

4. Sum to infinity

$$m \cos \theta - \tfrac{1}{3} m^3 \cos 3\theta + \tfrac{1}{5} m^5 \cos 5\theta \ldots\ldots,$$

m being less than unity; and expand the result in powers of $\cos \theta$.

5. Sum to infinity

$$\cos \theta + \frac{1}{2} \frac{\cos 3\theta}{3} + \frac{1.3}{2.4} \frac{\cos 5\theta}{5} + \frac{1.3.5}{2.4.6} \frac{\cos 7\theta}{7} + \ldots\ldots.$$

6. Prove that

$$\begin{vmatrix} 1, & 1, & 1 \\ \cos \alpha, & \cos \beta, & \cos \gamma \\ \cos^2 \alpha, & \cos^2 \beta, & \cos^2 \gamma \end{vmatrix}$$

is a factor in

$$\begin{vmatrix} \sin \alpha, & \sin \beta, & \sin \gamma \\ \sin 2\alpha, & \sin 2\beta, & \sin 2\gamma \\ \sin 4\alpha, & \sin 4\beta, & \sin 4\gamma \end{vmatrix}.$$

EXERCISE LXXIX.

1. If $A + B + C = 90^\circ$, prove that

$$\tan^4 A + \tan^4 B + \tan^4 C$$
$$= (\tan A + \tan B + \tan C)(\tan^3 A + \tan^3 B + \tan^3 C) + 2$$
$$- (\tan A + \tan B + \tan C) \sec A \sec B \sec C.$$

2. Prove that

$$\tfrac{1}{2}\theta = \sin \theta - \tfrac{1}{2} \sin 2\theta + \tfrac{1}{3} \sin 3\theta - \ldots,$$

3. Express

$$\log\sin(\alpha+\beta\sqrt{-1}) \text{ in the form } p+q\sqrt{-1}.$$

4. Sum to n terms

$$\frac{\sin\theta}{\cos\theta\cos 2\theta}+\frac{\sin 2\theta}{\cos\theta\cos 2\theta\cos 3\theta}+\frac{\sin 3\theta}{\cos 2\theta\cos 3\theta\cos 4\theta}+\dots.$$

5. Shew that

$$\cos\frac{\alpha}{n}\cos\frac{\alpha+2\pi}{n}\cos\frac{\alpha+4\pi}{n}\dots\cos\frac{\alpha+2(n-1)\pi}{n}$$

$$=\frac{(-1)^{\frac{n}{2}}-\cos\alpha}{2^{n-1}}$$

when n is even. Find its value when n is odd.

6. Prove that in any formula connecting the sides and angles of a triangle we may replace the sides by $a\cos A\cos 2A\cos 2^2 A\dots\cos 2^{n-1}A$, &c., and the angles by $p\pi+(-2)^n A$, $q\pi+(-2)^n B$, $r\pi+(-2)^n C$, where p, q, and r are any *odd* integers.

As an instance, prove that $-\cos 2^n C$ is equal to

$$\frac{(a\cos A\cos 2A\dots\cos 2^{n-1}A)^2+(b\cos B\cos 2B\dots\cos 2^{n-1}B)^2-(c\cos C\cos 2C\dots\cos 2^{n-1}C)^2}{2\,(a\cos A\cos 2A\dots\dots\cos 2^{n-1}A)\,(b\cos B\cos 2B\dots\dots\cos 2^{n-1}B)}.$$

EXERCISE LXXX.

1. Prove that

$$x^{27}-6x^9+12x^3-8x+1$$

$$=\frac{1}{\lfloor 1}(3-2)^3\log x+\frac{1}{\lfloor 2}(3^2-2)^3(\log x)^2$$

$$+\frac{1}{\lfloor 3}(3^3-2)^3(\log x)^3+\dots\dots \textit{ad inf.}$$

2. Prove that in any triangle

$$\frac{b^7\cos^7 C+c^7\cos^7 B-a^7}{b^5\cos^5 C+c^5\cos^5 B-a^5}=\frac{7}{12}(b^2\cos^2 C+c^2\cos^2 B+a^2)^2.$$

3. Sum to n terms

$$1+\frac{\cos\alpha}{\cos\alpha}+\frac{\cos 2\alpha}{\cos^2\alpha}+\frac{\cos 3\alpha}{\cos^3\alpha}+\ldots,$$

and shew that the sum vanishes when $\alpha=\dfrac{\pi}{n}$.

4. If n be the number of seconds in a small angle θ, prove that, approximately,

$$\log n=L\tan\theta+\tfrac{2}{3}L\cos\theta-11{\cdot}3522412.$$

5. If $\cos(\alpha+\beta\sqrt{-1})=\cos\phi+\sqrt{-1}\sin\phi$, we shall have

$$\sin\phi=\pm\sin^2\alpha=\pm\frac{(e^\beta-e^{-\beta})^2}{4}.$$

6. Prove that

$$\begin{vmatrix} 1, & \cos(\alpha-\beta), & \cos(\alpha-\gamma) \\ \cos(\beta-\alpha), & 1, & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha), & \cos(\gamma-\beta), & 1 \end{vmatrix}=0.$$

7. Shew that $\dfrac{\sin x+\sin y}{\sin y}$

$$=\left(1+\frac{x}{y}\right)\left(1+\frac{x}{\pi-y}\right)\left(1-\frac{x}{\pi+y}\right)\left(1+\frac{x}{2\pi+y}\right)$$

$$\times\left(1-\frac{x}{2\pi-y}\right)\left(1+\frac{x}{3\pi-y}\right)\left(1-\frac{x}{3\pi+y}\right)\ldots\ldots$$

ANSWERS.

I.

1. $\frac{200}{9}\pi$ miles.

2. $\frac{\pi}{10}$, $\frac{5}{18}$, $\frac{\pi}{18}$, $\frac{2}{3}$.

3. $57^\circ\, 17'\, 44''\cdot 8$ nearly.

4. $60^g\, 24^{`}\, 72^{``}\cdot\dot{2}$.

5. 6°.

6. $\frac{5}{19}\pi$, $\frac{9}{38}\pi$, $\frac{\pi}{2}$.

II.

1. $172^\circ\, 30'$.

2. $7^\circ\, 12'$, 3°, $55^g\, 25^{`}$.

3. -1, 0, 1, 1, -1.

5. $\frac{1}{\sqrt{5}}$, $\frac{3}{5}$, $\frac{15}{4}$, $\frac{21}{5}$.

III.

1. (1) $\frac{1}{2}-\frac{1}{\sqrt{3}}$. (2) 5. (3) 0.

2. When the unit is 18°, $m=\frac{10}{\pi}$.

3. 12π inches, 144°.

4. (1) sec 34°. (2) cosec 5°. (3) vers 20°.

5. $168^\circ\, 45'$, $187^g\, 50^{`}$, $\frac{15}{16}\pi$.

IV.

2. (1) $\pm\frac{63}{16}$, $\pm\frac{33}{56}$. (2) $\pm\frac{56}{33}$, $\pm\frac{16}{63}$. (3) $\pm\frac{24}{25}$. (4) $\frac{63}{65}$, $\pm\frac{33}{65}$. (5) $\frac{56}{65}$, $-\frac{16}{65}$.

3. (1) $\frac{63}{16}$. (2) $\frac{56}{33}$. (3) $\frac{24}{25}$. (4) $\pm\frac{63}{65}$. (5) $\pm\frac{56}{65}$.

5. 61·56 seconds.

V.

2. $146^g\cdot\dot{6}$, $\dfrac{39}{65}\pi$.

3. (1) $\dfrac{\sqrt{3}}{2}$. (2) $\dfrac{\sqrt{3}}{2}$.

VI.

1. $1^\circ 48'$.

3. (1) $\tan 2a$. (2) $\tan\theta$.

4. Sin 18°, $-\cos 18^\circ$, $-\cot 18^\circ$; $-\sin 36^\circ$, $-\cos 36^\circ$, $\cot 36^\circ$; $-\sin 36^\circ$, $\cos 36^\circ$, $-\cot 36^\circ$.

5. $9^\circ\ 32'\ 57''$ nearly.

VII.

2. From the given values we have either $B = \pm C$ or $B = \pm(180^\circ - C)$; therefore we easily obtain the values $\pm\dfrac{5}{13}$, $\pm\dfrac{9875}{13\times 29^2}$, $\pm\dfrac{10285}{13\times 29^2}$.

4. $82^\circ\ 30'$, $91^g\cdot\dot{8}$, $\frac{11}{24}\pi$.

VIII.

1. (1) $-\cot 2\theta$. (2) $\sin 6a$. (3) $\tan^2\phi$.

5. 5 miles, 1030 yards, nearly.

IX.

1. $\pm\dfrac{\sqrt{5}}{3}$, $\pm\dfrac{2}{3}$, $\dfrac{4\sqrt{5}}{9}$, $-4\sqrt{5}$; $\pm\dfrac{2}{\sqrt{5}}$ or $\pm\dfrac{1}{\sqrt{5}}$, 2 or $\dfrac{1}{2}$.

Tan $\dfrac{B}{2}$ must be found from the formula $\sin B = \dfrac{2\tan\frac{B}{2}}{1+\tan^2\frac{B}{2}}$. If we use the equation $\dfrac{2\tan\frac{B}{2}}{1-\tan^2\frac{B}{2}} = \tan B = \pm\frac{4}{3}$ we shall obtain four roots, of which the two negative ones are extraneous, since $\sin B$ and $\tan\dfrac{B}{2}$ have always the same sign.

X.

2. $\dfrac{n\pi}{1-\dfrac{\pi}{180}}$.

XI.

1. $\theta = m\pi + \dfrac{\pi}{8}$ where m is zero, or any integer positive or negative. Bring the equation into the form $\sin\left(\dfrac{\pi}{4}-\theta\right) = \sin\theta$.

3. 521 miles, 1502 yards, nearly.

XII.

1. (1) $2n\pi \pm \dfrac{\pi}{3}$ or $n\pi$. (2) $\dfrac{2}{3}n\pi + \dfrac{\pi}{6}$ or $(2n+1)\pi$.

5. The Arithmetic means are $29^\circ 5' 11''$ nearly; $32^g 37' 37''$ nearly; $\frac{19}{3600}\pi$. The Harmonic means are $1^\circ 46' 20''$ nearly; $2^g 18' 41''$ nearly; $\dfrac{\pi}{190}$.

XIII.

1. $n\frac{\pi}{2}+\frac{\pi}{4}$.

5. $\frac{1}{12}$.

XIV.

2. (1) $n\frac{\pi}{5}+\frac{\pi}{20}$ or $n\frac{\pi}{2}-\frac{\pi}{8}$. (2) $\frac{2}{3}n\pi\pm\frac{\pi}{18}$ or $n\frac{\pi}{3}$.

3. (1) $\pm\frac{np-mq}{\sqrt{(p^2+q^2)(m^2+n^2)}}$. (2) $\pm\frac{m(q^2-p^2)+2npq}{(p^2+q^2)\sqrt{m^2+n^2}}$.

(3) $\frac{np+mq}{nq-mp}$.

5. $n\pi+\frac{\pi}{6}$.

XV.

1. (1) $n\pi+\frac{\pi}{8}$. (2) $n\pi+\frac{\pi}{4}$.

5. $\pm\frac{2}{\sqrt{13}}$, $\pm\frac{12}{13}$, $\pm\frac{120}{169}$.

XVI.

1. $\frac{5453}{15625}$.

2. (1) $2n\frac{\pi}{3}+\frac{\pi}{6}$ or $2n\pi$. (2) $n\frac{\pi}{3}\pm\frac{\pi}{18}$ or $n\frac{\pi}{3}$.

4. $A=240°$, $B=180°$.

6. $2\sqrt{3}$ inches, 1 inch.

XVII.

1. $\pm\frac{3}{\sqrt{58}}$ or $\pm\frac{7}{\sqrt{58}}$; $\frac{3}{7}$ or $-\frac{7}{3}$.

3. (1) $(2n+1)\frac{\pi}{2}$ or $2n\frac{\pi}{5}+\frac{\pi}{10}$. (2) $n\pi\pm\frac{\pi}{6}$ or $n\pi$.

XVIII.

3. $n\frac{\pi}{2} \pm \frac{\pi}{24}$ or $n\pi + \frac{\pi}{2}$.

XIX.

1. $2n\frac{\pi}{3} \pm \frac{\pi}{15}$.

4. 3; -2.

XX.

1. $a^2\{y^2 + (x+b)^2\} = \{y^2 + x^2 - b^2\}^2$. Find the values of $x\cos\theta + y\sin\theta$ and $x\sin\theta - y\cos\theta$ respectively, and thence eliminate θ.

2. $n\pi + \frac{\pi}{4}$ or $n\frac{\pi}{2} + (-1)^n\frac{\pi}{12}$.

3. Use the formula $\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x}$.

XXI.

3. (1) $\frac{(2n-1)\pi}{k+l}$ or $\frac{2n\pi}{k-l}$.

(2) $\frac{(4n+1)\pi}{2(k+l)}$ or $\frac{(4n-1)\pi}{2(k-l)}$. (3) $\frac{(2n+1)\pi}{2(k+l)}$.

4. $\begin{cases} \theta = k\pi + \frac{\pi}{3} \\ \phi = l\pi + \frac{\pi}{6} \end{cases}$ or $\begin{cases} \theta = \frac{2k+1}{2}\pi + \frac{\pi}{6} \\ \phi = \frac{2l+1}{2}\pi + \frac{\pi}{3}. \end{cases}$

6. 30°, 60°, 90°.

XXII.

3. $\alpha = (m+n)\frac{\pi}{2} + \frac{\pi}{4}$, $\beta = (m-n)\frac{\pi}{2} + \frac{\pi}{12}$.

4. $\phi = n\frac{\pi}{4}$.

XXIII.

2. (1) $n\pi \pm \frac{\pi}{12}$. (2) $2n\pi + \frac{\pi}{4}$ or $2n\frac{\pi}{3} + \frac{\pi}{4}$.
3. As 9 is to 1.
5. 2·6074550, $\bar{1}$·0107239, 3·2037341.
6. $2x = (x^2 + y^2)^2 - 3(x^2 + y^2)$.

XXIV.

2. (1) $n\pi + \frac{\pi}{4}$ or $n\pi + \alpha$, where $\tan \alpha = 2$.

   (2) $n\pi + (-1)^n \frac{\pi}{6}$ or $n\pi + (-1)^n \beta$, where $\sin \beta = \frac{1}{4}$.

XXV.

1. 60°.
2. (1) $n\frac{\pi}{2} + \frac{\pi}{4}$. (2) $n\frac{\pi}{4} + \frac{\pi}{8}$, $n\pi + \frac{3\pi}{8}$, $n\frac{\pi}{3} + \frac{\pi}{8}$.
3. $(m^2 - n^2)^2 = 16\,mn$.
6. 1 mile, 472 yards, nearly.

XXVI.

2. 3·605.
4. (1) $n\pi \pm \frac{5\pi}{12}$, $n\pi \pm \frac{\pi}{12}$.

   (2) $n\pi - (-1)^n \frac{5\pi}{12}$, $n\pi - (-1)^n \frac{\pi}{12}$.
6. $e = 2·718281$.......

XXVII.

1. There are two solutions:

   $A = 30^\circ$, $B = 23^\circ\,35'$ nearly, $C = 126^\circ\,25'$ nearly,
   $b = 8$, $c = 16·0937$ nearly;

or $A=150°$, $B=23° 35'$ nearly, $C=6° 25'$ nearly,
$b=8$, $c=2\cdot2375$ nearly.

2. $\frac{4}{11}\sqrt{14}, 9\sqrt{2}, 10$. Use the formula $a^2=b^2+c^2-2bc\cos A$.

5. $\theta=n\pi\pm\frac{\pi}{3}$ or $n\pi+\frac{\pi}{2}\pm\frac{\alpha}{2}$, where $\cos\alpha=\frac{7}{10}$.

XXVIII.

5. As $4-2\sqrt{3}$ is to 1.

XXIX.

3. $8, 4+2\sqrt{3}$.

4. (1) $2n\pi\pm(\alpha+\beta)$ or $2n\pi\pm(\alpha-\beta)$.

(2) $n\pi+\frac{\pi}{4}$ or $n\frac{\pi}{2}+\gamma$, where $\tan 2\gamma=2$.

XXXI.

1. $\sin^3\theta\cos^5\theta=-\frac{1}{2^7}\sin 8\theta-\frac{1}{2^6}\sin 6\theta+\frac{1}{2^6}\sin 4\theta+\frac{3}{2^6}\sin 2\theta$.

3. $\cdot383$ nearly.

6. $\begin{cases} x=\dfrac{m+n}{3}\pi+\dfrac{\pi}{6} \\ y=(m-n)\pi \end{cases}$ $\begin{cases} x=\dfrac{3n-m}{4}\pi+\dfrac{\pi}{8} \\ y=\dfrac{3m-n}{4}\pi-\dfrac{\pi}{8} \end{cases}$

$\begin{cases} x=(m-n)\pi \\ y=\dfrac{m+n}{3}\pi-\dfrac{\pi}{6} \end{cases}$ $\begin{cases} x=\dfrac{3m+n}{4}\pi+\dfrac{\pi}{8} \\ y=\dfrac{3n+m}{4}\pi-\dfrac{\pi}{8}. \end{cases}$

XXXII.

2. $B=144°$, $C=18°$, $b=2\sqrt{10+2\sqrt{5}}$, $c=4$.

3. $\{2m+1-(m-n)^2\}^2+\{2n+1-(m-n)^2\}^2=4$.

4. $2\cdot5649494$.

XXXIII.

6. The error in C is approximately equal to $a\theta \sin B$, where θ is the error in the measurement of C.

XXXIV.

1. $x^2 - \frac{\sqrt{5}}{2}x + \frac{1}{4} = 0.$

2. 880·83, 1297·79, and 1490·35 nearly. Use a table of logarithms.

XXXV.

3. $\begin{cases} \theta = (n-r)\pi + \dfrac{\alpha-\beta}{2} \\ \phi = (n+r)\pi - \dfrac{\alpha+\beta}{2} \end{cases} \quad \begin{cases} \theta = (n-r)\pi + \dfrac{\pi}{2} - \dfrac{\alpha-\beta}{2} \\ \phi = (n+r)\pi + \dfrac{\pi}{2} - \dfrac{\alpha+\beta}{2} \end{cases}$

4. (1) $n\pi \pm \frac{\pi}{3}$, $n\frac{\pi}{2} + \frac{\pi}{8}$, $n\frac{\pi}{2}$.

(2) $n\pi + \tan^{-1}\frac{3\sin^2\alpha - 1}{\sin\alpha\cos\alpha}$, $n\pi + \tan^{-1}\frac{\sin^2\alpha + 1}{\sin\alpha\cos\alpha}$.

XXXVI.

1. x is equal to the diameter of the circumscribing circle.
2. $180^\circ\, 49'\, 56''$ nearly, or $61^\circ\, 10'\, 4''$ nearly.
4. m^2 lies between $\frac{3}{4}$ and 1.

XXXVII.

5. $n\frac{\pi}{2} + \frac{\pi}{12}$.

XXXVIII.

1. $\pm\frac{4}{5}$.
6. $\beta^2 = \alpha(\alpha + \beta + \gamma)$.

XXXIX.

2. 251·92.

6. (1) $2n\pi + \frac{\pi}{2}$. (2) $n\pi$ or $n\pi \pm \frac{\pi}{3}$.

XL.

3. See page 47.

XLI.

2. The greatest value is $\frac{5}{3}$.

5. 777, 1942·5, 868·7.

XLII.

2. The sides will be 2079, 4347, and 5481.

6. ·16854.

XLIII.

2. (1) $\sqrt{6} + \sqrt{2}$, $\sqrt{6} - \sqrt{2}$.

   (2) 0 or $\pm \sqrt{3}$. (3) $\frac{1}{4} \pm \frac{1}{2\sqrt{6}}$.

XLIV.

2. 2·1095 nearly.

3. $ab^2 = 2c\,(b^2 - a^2)$.

XLV.

3. $A = 120°$, $B = C = 30°$, $a = 2\sqrt{3}$, $b = c = 2$.

4. 16·68 inches.

5. $\frac{R}{r} = \frac{5}{2}$.

XLVI.

5. $a \cot A \sin B \sin C$, $b \cot B \sin C \sin A$, $c \cot C \sin A \sin B$.

7. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4^{\frac{2}{3}}$.

XLVII.

2. $r = 2n - 1$.

XLIX.

5. Let d be the length of the diameter, and let the diagonals intersect in Q. Then

$$\begin{aligned} AB &= d\cos\beta, & BC &= d\sin\alpha, \\ CD &= d\sin\beta, & DA &= d\cos\alpha, \\ AC &= d\cos(\alpha-\beta), & BD &= d\sin(\alpha+\beta), \\ AQ &= d\cos\alpha\cos\beta, & QC &= d\sin\alpha\sin\beta, \\ BQ &= d\sin\alpha\cos\beta, & QD &= d\cos\alpha\sin\beta. \end{aligned}$$

6. 1·06258.

L.

1. They subtend an angle of 120° at the *internal* centre of similitude, and 60° at the *external* one.

3. Let the given angle $= \theta$. Then A is determined by the equation $\sin\frac{A}{2} = \frac{1}{2}\cos\theta \pm \sqrt{\frac{1}{4}\cos^2\theta - \frac{r}{2R}}$.

After A has been found, B and C are determined by

$$B + C = 180° - A, \quad B - C = 2\theta.$$

Then $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$.

LI.

1. The roots are $4\cos^2 10°$, $4\cos^2 50°$, $4\cos^2 70°$.
5. $0, \pm\frac{2}{3}$.
7. The area is $abcd\,(ac - bd)\,(bc + ad)$.

LII.

1. ·3780762.

6. (1) $x = 2\sqrt{a}\cos\left(\theta \pm \frac{\pi}{4}\right)$, where $b = a\cos 2\theta$.

(2) $x = 2\operatorname{cosec} 2\theta$ or $2\cot 2\theta$, where $b = a\cos 2\theta$.

(3) $\sin(x + \phi) = \frac{c}{a}\cos\phi$, where $b = a\tan\phi$.

LIII.

2. $\cdot 00068$.

3. $x = 0$. If $m = n$, x may also $= \pm \frac{1}{m}$.

6. $\sin \phi = \frac{(1 + \sin \psi)^{\frac{1}{3}} - (1 - \sin \psi)^{\frac{1}{3}}}{(1 + \sin \psi)^{\frac{1}{3}} + (1 - \sin \psi)^{\frac{1}{3}}}$.

LIV.

2. $n\pi \pm \frac{\pi}{6}$.

6. $\cos 2\eta + \cos 2\zeta = \frac{2(p^2 - q^2)(1 - 2p^2 - 2q^2)}{p^2 + q^2}$.

LV.

2. $\pm \frac{c^2}{2} \operatorname{cosec} C \sin (A - B)$.

LVI.

1. $C = 45^\circ$ or 135°.

5. $m\pi + \tan^{-1} \frac{\sin (\alpha - \beta)}{2 \sin \alpha \sin \beta}$.

LVII.

1. $\frac{R}{r} = 2\cdot 637$ nearly. Use the table of logarithms and the formula $\frac{r}{R} + 1 = \cos A + \cos B + \cos C$.

4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

5. See page 44, remembering that ABC is the *pedal triangle* of $I_1 I_2 I_3$.

LVIII.

1. $\tan^{-1} \dfrac{36}{37}$.

3. $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$.

4. $\dfrac{n\pi}{4} - \dfrac{a}{4} + (-1)^{n+1} \dfrac{1}{4} \sin^{-1} \{3 \sin a\}$.

5. The area is $\Delta \left(1 + 2\dfrac{r_a}{p_a} + 2\dfrac{r_b}{p_b} + 2\dfrac{r_c}{p_c}\right)^2$. See page 8.

LIX.

1. 11.
2. $1 + 2\sqrt{3} \cos 50°$, $1 + 2\sqrt{3} \cos 70°$, $1 + 2\sqrt{3} \cos 170°$.
3. $a = 10 \pm 3\sqrt{5}$, $b = 10 \mp 3\sqrt{5}$.

LX.

1. ·288.
3. $\rho^2 - a\rho \cos \theta = 2a^2$.

LXI.

2. 22·60962.

4. $m = \dfrac{\tan \dfrac{\phi' - \phi''}{2}}{\tan \dfrac{\phi' + \phi''}{2}}$ or $\dfrac{\cot \dfrac{\phi' - \phi''}{2}}{\cot \dfrac{\phi' + \phi''}{2}}$.

5. $\frac{32}{625}$.

LXII.

2. $r = 2·5$.

4. $\dfrac{(n+1) \sin nx - n \sin (n+1) x}{2(1 - \cos x)}$.

LXIII.

3. $\dfrac{a\beta \sin A \cos B + b\alpha \sin B \cos A}{c}$.

4. The product is $\left\{\cos\dfrac{\pi-\theta}{p} - \sqrt{-1}\sin\dfrac{\pi-\theta}{p}\right\}^2$.

LXIV.

1. The value is 1.
4. $\log(x \pm \sqrt{x^2-a^2}) - \log a$.
5. $c^2\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right) = \left(4 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}\right)\left(\dfrac{a^2y^2}{b^2}+\dfrac{b^2x^2}{a^2}\right)$.
6. $\sin\dfrac{2\pi}{n+2}$.

LXV.

2. $\dfrac{m}{3}$.

3. $\cos\alpha\dfrac{e^b+e^{-b}}{2} + \sqrt{-1}\sin\alpha\dfrac{e^b-e^{-b}}{2}$.

4. $\theta = 2n\pi$, or $\theta = 2m\pi \pm \cos^{-1}\left\{\dfrac{a}{b}\cos\alpha\right\}$, where a is greater than b.

LXVI.

2. 15·16571, 12·44989, 7·07106.
3. $\theta = \left(r + \dfrac{1}{4}\right)\dfrac{\pi}{n}$.

LXVII.

2. $lmnr = \dfrac{1}{64\cos^2 A\cos^2 B\cos^2 C}$.

LXVIII.

1. $-1+2\cos 6^\circ$, $-1-2\cos 54^\circ$, $-1-2\cos 66^\circ$.

4. $\log \dfrac{1}{2\sin\frac{\theta}{2}}$.

5. $\dfrac{2\cos a\,(e^{\beta}+e^{-\beta})}{e^{2\beta}+e^{-2\beta}+2\cos 2a}+\sqrt{-1}\,\dfrac{2\sin a\,(e^{\beta}-e^{-\beta})}{e^{2\beta}+e^{-2\beta}+2\cos 2a}$.

LXIX.

1. When n is of the form $4r$ $\quad \theta=2m\pi\pm cx$,

" " " $4r+1$ $\quad \theta=2m\pi\pm\left(\frac{\pi}{2}+cx\right)$,

" " " $4r+2$ $\quad \theta=(2m+1)\pi\pm cx$

" " " $4r+3$ $\quad \theta=2m\pi\pm\left(\frac{\pi}{2}-cx\right)$.

2. $\pm\frac{3}{2}$ or $\pm\dfrac{\cos^2\phi}{2+\sin^2\phi}$.

3. $\dfrac{(1-2n)\cos n\theta+(1+2n)\cos(n-1)\theta-\cos\theta-1}{2(1-\cos\theta)}$.

4. $\dfrac{\pi^2}{6}+\dfrac{1-\frac{\pi^2}{6}}{m^2}$.

6. $\dfrac{1}{2}\left\{5\dfrac{\sin\tau}{\sin\sigma}-1\right\}$.

LXX.

1. The formula may be simplified by the following substitutions.

Put $a=2\theta-\frac{\pi}{4}$, $\beta=2\phi-\frac{\pi}{4}$, and $\gamma=2\psi-\frac{\pi}{4}$.

Then $$2(\theta+\phi+\psi)-3\frac{\pi}{4}=\frac{\pi}{4},$$

or $$\theta+\phi+\psi=\frac{\pi}{2}.$$

2. The value is 2.

3. Remember that $x=e^{\log x}$.

LXXI.

1. $\sqrt[4]{P}=\sin(\theta+\phi)$.

4. Let S_1 be the sum of the first series, and S_2 that of the second; then

$$S_1+\sqrt{-1}\,S_2=e^{a\sqrt{-1}}\,e^{xe^{\beta\sqrt{-1}}},$$

and $\therefore$ (1) $S_1=e^{x\cos\beta}\cos(a+x\sin\beta)$,

(2) $S_2=e^{x\cos\beta}\sin(a+x\sin\beta)$.

LXXII.

1. Put $\theta=18\phi-\pi$.

2. The limiting value is *infinite.*

3. Let a, b, c, d, ... be the lengths of the sides, and let the angle between any two sides a and b be denoted by $\widehat{ab}$; then

$$a^2=b^2+c^2+d^2+e^2+\ldots\ldots-2bc\cos\widehat{bc}-2cd\cos\widehat{cd}$$
$$-2bd\cos\widehat{bd}-2be\cos\widehat{be}-\ldots\ldots.$$

4. $\dfrac{1}{y^2}=\dfrac{1}{b^2}-\dfrac{1}{a^2}$.

6. $\dfrac{\sin\dfrac{n\theta}{2}}{\sin\dfrac{\theta}{2}}\left\{\cos(n+1)\dfrac{\theta}{2}+\sqrt{-1}\sin(n+1)\dfrac{\theta}{2}\right\}$.

LXXIII.

2. $\cdot 054718$.

3. (1) -1. (2) $+1$.

4. $\frac{1}{2}\{e\cos\theta + e^{\cos 2\theta}\cos(3\theta + \sin 2\theta)\}$.

5. $$a\sin\alpha\frac{e^{\beta}+e^{-\beta}}{2} - b\cos\alpha\frac{e^{\beta}-e^{-\beta}}{2}$$
$$+\sqrt{-1}\left(b\sin\alpha\frac{e^{\beta}+e^{-\beta}}{2}+a\cos\alpha\frac{e^{\beta}-e^{-\beta}}{2}\right).$$

LXXIV.

4. The product to n factors is $\dfrac{\sin\theta}{2}\left(\dfrac{\sin\frac{\theta}{2^n}}{2}\right)^{-\frac{1}{2^n}}$. The product to infinity is $\dfrac{\sin\theta}{2}$.

LXXV.

2. $e^{\cot\theta}\cos(\theta+1)$.

3. The error is of the fifth order.

LXXVI.

3. $x = \pm\dfrac{2abc}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}}$. If a, b, and c are all positive and are the sides of a triangle, x is the diameter of the circumscribing circle.

5. See pages 75 and 76.

6.
$$\begin{vmatrix} a\tan\beta - b\tan\alpha, & (a-b)(1-\tan\alpha\tan\beta), & a\tan\alpha - b\tan\beta \\ b\tan\gamma - c\tan\beta, & (b-c)(1-\tan\beta\tan\gamma), & b\tan\beta - c\tan\gamma \\ c\tan\alpha - a\tan\gamma, & (c-a)(1-\tan\gamma\tan\alpha), & c\tan\gamma - a\tan\alpha \end{vmatrix} = 0.$$

LXXVII.

1. $a=\frac{1}{2}\left\{\left(c^2+4\Delta\cot\frac{C}{2}\right)^{\frac{1}{2}}+\left(c^2-4\Delta\tan\frac{C}{2}\right)^{\frac{1}{2}}\right\},$

$b=\frac{1}{2}\left\{\left(c^2+4\Delta\cot\frac{C}{2}\right)^{\frac{1}{2}}-\left(c^2-4\Delta\tan\frac{C}{2}\right)^{\frac{1}{2}}\right\}.$

2. $e^{\frac{3}{2}}$.

LXXVIII.

2. $n\pi+\frac{\pi}{8}\pm\frac{1}{2}\cos^{-1}\frac{23}{17\sqrt{2}}$.

4. The sum $=\frac{1}{2}\tan^{-1}\frac{2m\cos\theta}{1-m^2}$

$$=\frac{1}{2}\left\{\frac{2m\cos\theta}{1-m^2}-\frac{1}{3}\left(\frac{2m\cos\theta}{1-m^2}\right)^3+\ldots\ldots\right\}.$$

5. $\frac{1}{2}\cos^{-1}(2\sin\theta-1)$.

6. The first determinant

$$=8\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha+\gamma}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\beta+\gamma}{2}.$$

Now the values $\alpha=\pm\beta=\pm\gamma$ make the second determinant vanish, and hence it involves (as factors) sines of some multiples of these angles. Subtracting columns we see at once that $\frac{\alpha-\beta}{2}$ is the least multiple of $\alpha-\beta$ that occurs; and similarly for the other factors.

LXXIX.

3. $p=\frac{1}{2}\log\{e^{2\beta}+e^{-2\beta}-2\cos 2\alpha\}-\log 2,$

$q=\cot^{-1}\left\{\frac{e^{\beta}+e^{-\beta}}{e^{\beta}-e^{-\beta}}\tan\alpha\right\}.$

4. $\dfrac{1}{2\sin\theta\cos n\theta\cos(n+1)\theta} - \dfrac{1}{\sin 2\theta}$.

6. **This is an extension of the method given on page 44.**

LXXX.

3. $\dfrac{\sin n\alpha}{\sin\alpha\cos^{n-1}\alpha}$.

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